Remind*

* Energy theorems:

*1, principle of virtual work (Due to virtual displacement)

One Particale: In variational mechanics, we imagine displacement to occur when in reality no such displacements exist. These fictitous displacement are called virtual displacements and the work done by these imaginary displacements is refred to as the virtual work.

The virtual displacements are assumed as small so that there will be no significant changes in geometry. Thus the force may also be assumed to remain unchanged during the virtual displacements.

Let us consider a particale undergoing a virtual displacement δq as shown in fig*1.a. The particale is subjected to a system of, say, three forces P_1, P_2 and P_3, respectively. The resulting virtual work can be expressed as

$$\delta W = P_1\,\delta q_1 + P_2\,\delta q_2 + P_3\,\delta q_3 \qquad (1)^*$$

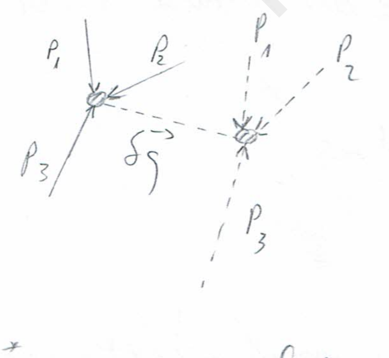

fig*1.a: Praticale in equilibrium undergoing a virtual displacement.

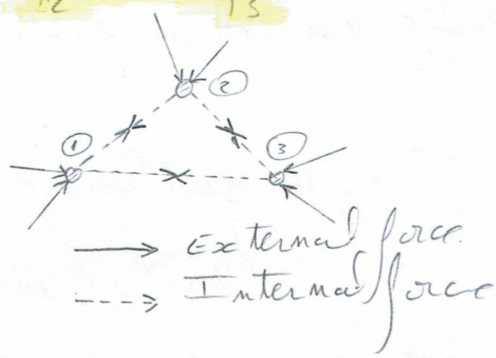

fig 1.b: three particles in equilibrium.

Where δ is a variational operator denoting virtual quantity and δq_1, δq_2 and δq_3 are the three components of the virtual displacement δq in the P_1, P_2, and P_3 direction, respectively.

because the three forces P_1, P_2 and P_3 are in equilibrium, their vectorial sum must vanish and so does the work done for a virtual displacement,

$$\delta W = 0 \quad \text{———} \quad (2)^*$$

Alternatively, if it is known that $\delta W = 0$ it follows that the vectorial sum of the force is zero and the forces are in equilibrium

<u>theorem</u>: A necessary and sufficient condition for the equilibrium of praticale is that the virtual work done by the forces acting on the particale vanishes for any virtual displacement.

A number of particales:

The internal forces of an elastic continuum system represent the actions of the particles on each other and therefore occur as pairs of equal and opposite forces. This phenomenon is shown in fig 1.b for a three particles,

$$\sum_{j=1}^{3} \left(\overset{\text{external}}{\delta W_j^e} + \overset{\text{internal}}{\delta W_j^i} \right) = 0 \quad \text{———} \quad (3)^*$$

Where:

δW_j^e : virtual work of external forces for pratical;

δW_j^i : virtual work of internal forces for partical;

For an elastic continuum system, there are infinite nuber of particles.

Equation (3)* may be further generalized as

$$\delta W^e + \delta W^i = 0 \quad \text{———} \quad (4)^*$$

Where W is, for simplicity, defined as the summation of the work for the system of infinite nuber of particales.

<u>Continuum (elastic structure)</u>. A continuum is defined as a domain in which matter exists at every point. we may think of continuum as consisting of an infinite number of particales.

Theorem: A necessary and sufficient condition for the equilibrium of a system of particule or a continuum system is that the virtual work done by the external forces plus the virtual work done by the internal forces vanish for any virtual displacement.

For an elastic continuum system or an elastic structure, the work done by the internal forces W^i is equal and opposite to the strain energy U stored in the system during the deformation:

$$W^i = -U \quad \text{——— } (5)^*$$

Substituting Eq $(5)^*$ in Eq $(4)^*$ and simply using W to represent the external W^e results in

$$\delta W = \delta U \quad \text{——— } (6)^*$$

which states that external virtual work done due to a virtual displacement is equal to the internal virtual strain energy caused by the same virtual displacement.

This statement is the principle of virtual work (due to virtual displacement).

2. Castigliano's First theorem

Let us consider a structure subjected to a set of n external forces $P_1, P_2, \ldots, P_i, \ldots, P_n$ and apply a virtual displacement δq. This virtual displacement is allowed to take place in the structure in such a manner that δq is continuous everywhere but vanishes at all points of loading except under P_i. Thus only the δq absorb work from P_i; the other external forces do not do any work.

Due to δq_i, the virtual work done is $P_i \delta q_i$. According to the principle of virtual work as described in eq (6^*) this virtual work equals the virtual strain energy:

$$\delta U = P_i \, \delta q_i$$

or in the limit,

$$P_i = \frac{\partial U}{\partial q_i} \quad \text{——— } (7.a)^*$$

If instead of virtual displacement δq, we introduce a virtual rotation $\delta \theta_i$ at the point i where a moment M_i is acting, Eq (7.a) can be written as:

$$M_i = \frac{\partial U}{\partial \theta_i} \quad \text{---} \quad (7.b^*)$$

Equation (7.a) and (7.b) are known as Castigliano's first theorem, After Alberto Castigliano, who published the theorem in his Thesis for the engineer's degree in 1879. It is noted that this theorem is the fondation for the derivation of the finite element stiffness equations, especially for sophisticated elements for which the stress-strain equilibrium approch becom difficult. It is also important to note that this theorem also applies when the structural behavior is nonlinear.

3) Principal of Complementary Virtual Work:

figure 2* shows a load-displacement relation curve. We define W^* and U^* as the complementary work and complementary strain energy, respectivly. Again δ is variational operator denoting a virtual quantity. From this figure we see that

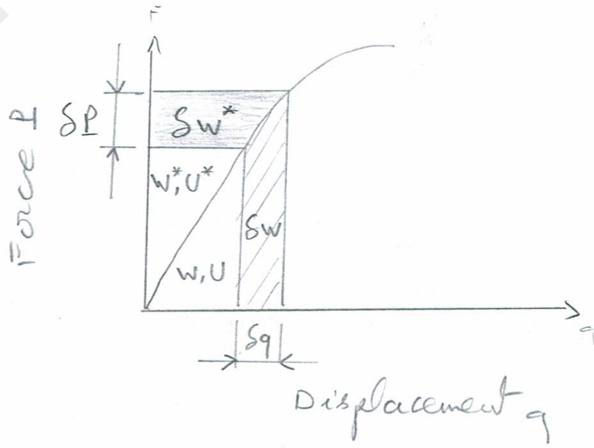

figure 2* Load-displacement curve and the associated works and energies.

$$\delta W^* = q\, \delta P \quad \text{———} \quad (8^*)$$

the principle of virtual work also for the case of complementary work and strain energy:

$$\delta W^* = \delta U^* \qquad (9^*)$$

In this case the virtual work is that due to a virtual force.

4. Castigliano's Second Theorem

Let us $P_1, P_2, \ldots P_i, \ldots P_n$. if a virtual increment δP_i is given to external static force P_i, the complementary work will be increased by δW^*. The only force that does any complementary work is P_i because other forces are not changed thus the increase in complementary energy can be obtained, following Eq (8^*), as:

$$\delta W^* = q_i \, \delta P_i \qquad (10^*)$$

Based on the principle of complementary virtual work as given in Eq (9^*)

$$\delta U^* = q_i \, \delta P_i \qquad (11^*)$$

Assuming the structural behavior to be linear, the load-displacement curve in fig 2^* becomes a straight line then:

$$\delta U = \delta U^* = q_i \, \delta P_i \qquad (12.a^*)$$

In the limit, Eq ($12^*.a$) becomes

$$q_i = \frac{\partial U}{\partial P_i} \qquad (12.b^*)$$

Wich is known as Castigliano's second theorem.

For linear structural behavior, the principle of superposition can be used. Equation (12.b), can, alternatively, be proven by applying δP_i first and after ward all the external forces. the load δP_i first produces an infinitesimal displacement so that the corresponding work is a small quantity of second order and can be neglected. When all the forces $P_1, P_2, P_3 \ldots$ are then applied, the work (or strain energy) in addition to that produced by all the forces is:

$$\delta U = q_i \, \delta P_i \qquad (12.a^*)$$

Which arrives at the same conclusion as given in Eqs ($12.a^*$) and ($12.b^*$).

If we consider a moment M_i and rotation θ_i instead of P_i and q_i, respectively, Eq (12.b) can be generalized to

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad (12.c)$$

In applying Castigliano's theorems, we must be careful that:

1- the displacement q_i (or rotation θ_i) must be referred to the same location at which the concentrated force P_i (or moment M_i) is acting.

2- q_i (or θ_i) must be referred to the same direction as P_i (or M_i)

3- The strain energy U must be formulated for entire structure based on all the external loads.

In some literature, Castigliano's first theorem is often referred to as the theorem of virtual work and Castigliano's second theorem is often simply referred to as Castigliano's theorem, and on occasion, as Castigliano's first theorem. In that case, the principle of least work is referred to as Castigliano's second theorem.

Strain Energy Expressions:

Castigliano's first theorem is the foundation for deriving finite element stiffness matrices.

Castigliano's second theorem is very useful in finding displacements and analyzing statically indeterminate structures. Its application is demonstrated in this chapter by a variety of examples.

Before demonstrating examples we must introduce the strain energy expressions for several basic structural element. It is assumed that the reader is familiar with of derivation of these expressions.

For a beam in bending: $U = \int_0^l \frac{M^2 dx}{2EI}$ ——— (13.a*) *
poutre

For a beam in shear: $U = \int_0^l \frac{V^2 dx}{2GA}$ ——— (13.b*) *
poutre

For a bar in tension: $\bar{U} = \int_0^l \frac{S^2\,dx}{2EA}$ (13.c*) x

For a bar in torsion: $\bar{U} \int_0^l \frac{T^2\,dx}{2GJ}$ (13.d*) *

Moment de torsion

For a panel in shear: $\bar{U} = \iint_{Area} \frac{q^2\,dx\,dy}{2Gt}$ (13.e*)

Where
- M = bending moment
- E = Modulus of Elasticity
- I = Moment of inertia
- V = shearing force
- $G = \frac{E}{2(1+\nu)}$, shear modulus
- A = cross-sectional area (m^2) or (mm^2)
- S = axial force
- T = twisting moment.