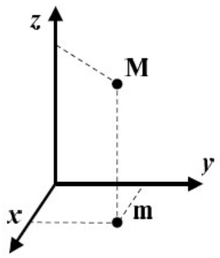
Opérateurs vectoriels

1. Calcule vectoriel

$$\begin{aligned}
\overrightarrow{\mathrm{rot}}\ \overrightarrow{\mathrm{grad}}\, V &= \vec{0} \\
\mathrm{div}\, \overrightarrow{\mathrm{rot}}\, \vec{A} &= 0 \\
\mathrm{div}\, \overrightarrow{\mathrm{grad}}\, V &= \Delta V \\
\overrightarrow{\mathrm{rot}}\ \overrightarrow{\mathrm{rot}}\, \vec{A} &= \overrightarrow{\mathrm{grad}}\, \mathrm{div}\, \vec{A} - \Delta \vec{A}
\end{aligned}$$

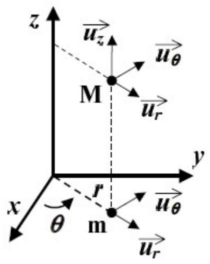
$$\begin{aligned}
\overrightarrow{\mathrm{grad}}(V_1 V_2) &= V_1\, \overrightarrow{\mathrm{grad}}\, V_2 + V_2\, \overrightarrow{\mathrm{grad}}\, V_1 \\
\overrightarrow{\mathrm{rot}}(V \vec{A}) &= V\, \overrightarrow{\mathrm{rot}}\, \vec{A} + \overrightarrow{\mathrm{grad}}\, V \wedge \vec{A} \\
\mathrm{div}(V \vec{A}) &= V\, \mathrm{div}\, \vec{A} + \overrightarrow{\mathrm{grad}}\, V \cdot \vec{A} \\
\mathrm{div}(\vec{A}_1 \wedge \vec{A}_2) &= \vec{A}_2 \cdot \overrightarrow{\mathrm{rot}}\, \vec{A}_1 - \vec{A}_1 \cdot \overrightarrow{\mathrm{rot}}\, \vec{A}_2
\end{aligned}$$

2. Coordonnées cartésiennes



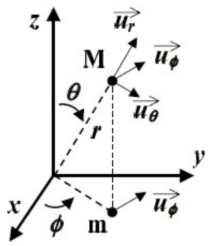
$$\begin{aligned}
\overrightarrow{\mathrm{grad}}\, V &= \frac{\partial V}{\partial x}\vec{u}_x + \frac{\partial V}{\partial y}\vec{u}_y + \frac{\partial V}{\partial z}\vec{u}_z \\
\mathrm{div}\, \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\overrightarrow{\mathrm{rot}}\, \vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{u}_z \\
\Delta V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}$$

3. Coordonnées cylindriques



$$\begin{aligned}
\overrightarrow{\mathrm{grad}}\, V &= \frac{\partial V}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u}_\theta + \frac{\partial V}{\partial z}\vec{u}_z \\
\mathrm{div}\, \vec{A} &= \frac{1}{r}\frac{\partial r A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\
\overrightarrow{\mathrm{rot}}\, \vec{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{u}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{u}_\theta + \frac{1}{r}\left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{u}_z \\
\Delta V &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}$$

4. Coordonnées sphérique



$$\begin{aligned}
\overrightarrow{\mathrm{grad}}\, V &= \frac{\partial V}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{u}_\phi \\
\mathrm{div}\, \vec{A} &= \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \\
\overrightarrow{\mathrm{rot}}\, \vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial \sin\theta A_\phi}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\vec{u}_r + \ldots \\
&\ldots + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial r A_\phi}{\partial r}\right)\vec{u}_\theta + \frac{1}{r}\left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{u}_\phi \\
\Delta V &= \frac{1}{r}\frac{\partial^2 r V}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}
\end{aligned}$$

Théorèmes

Théorème d'Ostrogradsky–Green :
S étant une surface fermée, τ le volume intérieur à S,

$$\oint_{(S)} \vec{A} \cdot \mathrm{d}\vec{S} = \int_{(\tau)} (\mathrm{div}\, \vec{A})\mathrm{d}\tau$$

Théorème de Stokes–Ampère :
C étant une courbe fermée bordant une surface S,

$$\oint_{(C)} \vec{A} \cdot \mathrm{d}\vec{l} = \int_{(S)} (\overrightarrow{\mathrm{rot}}\, \vec{A}) \cdot \mathrm{d}\vec{S}$$

Opération	Coordonnées cartésiennes (x,y,z)	Coordonnées cylindriques (ρ,φ,z)	Coordonnées sphériques (r,θ,φ)
Définition des coordonnées		$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$	$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$
$\boldsymbol{A}$	$A_x\boldsymbol{u}_x + A_y\boldsymbol{u}_y + A_z\boldsymbol{u}_z$	$A_\rho\boldsymbol{u}_\rho + A_\varphi\boldsymbol{u}_\varphi + A_z\boldsymbol{u}_z$	$A_r\boldsymbol{u}_r + A_\theta\boldsymbol{u}_\theta + A_\varphi\boldsymbol{u}_\varphi$
∇	$\boldsymbol{u}_x\frac{\partial}{\partial x} + \boldsymbol{u}_y\frac{\partial}{\partial y} + \boldsymbol{u}_z\frac{\partial}{\partial z}$	$\boldsymbol{u}_\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho}\boldsymbol{u}_\varphi\frac{\partial}{\partial\varphi} + \boldsymbol{u}_z\frac{\partial}{\partial z}$	$\boldsymbol{u}_r\frac{\partial}{\partial r} + \frac{1}{r}\boldsymbol{u}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\boldsymbol{u}_\varphi\frac{\partial}{\partial\varphi}$
$\nabla f \equiv \overrightarrow{\mathrm{grad}} f$	$\frac{\partial f}{\partial x}\boldsymbol{u}_x + \frac{\partial f}{\partial y}\boldsymbol{u}_y + \frac{\partial f}{\partial z}\boldsymbol{u}_z$	$\frac{\partial f}{\partial\rho}\boldsymbol{u}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\varphi}\boldsymbol{u}_\varphi + \frac{\partial f}{\partial z}\boldsymbol{u}_z$	$\frac{\partial f}{\partial r}\boldsymbol{u}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\boldsymbol{u}_\varphi$
$\nabla\cdot\boldsymbol{A} \equiv \mathrm{div}\,\boldsymbol{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial\rho A_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
$\nabla\wedge\boldsymbol{A} \equiv \overrightarrow{\mathrm{rot}}\boldsymbol{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{u}_z$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\boldsymbol{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\boldsymbol{u}_\varphi + \frac{1}{\rho}\left(\frac{\partial\rho A_\varphi}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\boldsymbol{u}_z$	$\frac{1}{r\sin\theta}\left(\frac{\partial A_\varphi\sin\theta}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\boldsymbol{u}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial rA_\varphi}{\partial r}\right)\boldsymbol{u}_\theta + \frac{1}{r}\left(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\boldsymbol{u}_\varphi$

$\Delta \boldsymbol{A} = \nabla^2 \boldsymbol{A}$	$\Delta A_x\, \boldsymbol{u}_x + \Delta A_y\, \boldsymbol{u}_y + \Delta A_z\, \boldsymbol{u}_z$	$\left(\Delta A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\boldsymbol{u}_\rho + \left(\Delta A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\boldsymbol{u}_\varphi + \Delta A_z\, \boldsymbol{u}_z$	$\left(\Delta A_r - \frac{2A_r}{r^2} - \frac{2A_\theta\cos\theta}{r^2\sin\theta} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\boldsymbol{u}_r + \left(\Delta A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\boldsymbol{u}_\theta + \left(\Delta A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\boldsymbol{u}_\varphi$

Quelques autres règles de calcul

$\operatorname{div}\ \overrightarrow{\operatorname{grad}} f = \nabla\cdot(\nabla f) = \nabla^2 f = \Delta f$ (laplacien)

$\overrightarrow{\operatorname{rot}}\ \overrightarrow{\operatorname{grad}} f = \nabla\wedge(\nabla f) = \mathbf{0}$

$\operatorname{div}\ \overrightarrow{\operatorname{rot}}\ \boldsymbol{A} = \nabla\cdot(\nabla\wedge\boldsymbol{A}) = 0$

$\overrightarrow{\operatorname{rot}}\ \overrightarrow{\operatorname{rot}}\ \boldsymbol{A} = \nabla\wedge(\nabla\wedge\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} = \overrightarrow{\operatorname{grad}}\ \operatorname{div}\boldsymbol{A} - \Delta\boldsymbol{A}$ (rotationnel du rotationnel)

$\Delta fg = f\Delta g + 2\nabla f\cdot\nabla g + g\Delta f$

Formule de Lagrange pour le produit vectoriel

$\boldsymbol{A}\wedge(\boldsymbol{B}\wedge\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$ si et seulement si A, B et C commutent entre eux.