# Methode de Cholesky

- A est dite symétrique si elle coincide avec sa transposée
ie : $A = A^T$ ou encore : $a_{ij} = a_{ji}$ $i,j = \overline{1,n}$
- A est dite définie positive si $\forall x \in \mathbb{R}^n$, $x \neq 0$,
ssi tous les mineurs principaux de A sont positifs $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0 \ldots \Delta_n > 0$

$$A = \begin{array}{c} \begin{array}{cccc} \Delta_1 & \Delta_2 & & \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \end{array}$$

La décomposition A en $S, S^t$ peut être faite directement à partir de la décomposition $A = LU$

En effet

$$A = LU = LIU$$

$$= LD^0U = \underbrace{LD^{1/2}}_{S}\underbrace{D^{-1/2}U}_{S^t}$$

$S^t = D^{-1/2}U$

$S = LD^{1/2}$

$$D = \begin{pmatrix} u_{11} & & 0 \\ & u_{22} & \\ 0 & & \ddots \\ & & & u_{nn} \end{pmatrix} \qquad D^{1/2} = \begin{pmatrix} \sqrt{u_{11}} & & 0 \\ & \sqrt{u_{22}} & \\ 0 & & \ddots & \\ & & & \sqrt{u_{nn}} \end{pmatrix}$$

ou en d'autre termes

$$S^t = \begin{pmatrix} \sqrt{u_{11}} & \frac{u_{12}}{\sqrt{u_{11}}} & \cdots & \frac{u_{1n}}{\sqrt{u_{11}}} \\ & \sqrt{u_{22}} & \cdots & \frac{u_{2n}}{\sqrt{u_{22}}} \\ & 0 & & \vdots \\ & & & \sqrt{u_{nn}} \end{pmatrix} \qquad S = \begin{pmatrix} L_{11}\sqrt{u_{11}} & & \\ L_{21}\sqrt{u_{11}} & L_{22}\sqrt{u_{22}} & \\ \vdots & \vdots & \ddots \\ L_{n1}\sqrt{u_{11}} & L_{n2}\sqrt{u_{22}} & \cdots & L_{nn}\sqrt{u_{nn}} \end{pmatrix}$$

$S_{ij} = \frac{u_{ij}}{\sqrt{u_{ii}}}$ ou bien $S_{ij} = L_{ij}\sqrt{u_{jj}}$

Exple :

$$A = \begin{pmatrix} 4 & 6 & 8 \\ 6 & 25 & 32 \\ 8 & 32 & 77 \end{pmatrix} \xrightarrow{u=} \underbrace{\begin{pmatrix} 4 & 6 & 8 \\ 0 & 16 & 20 \\ 0 & 0 & 36 \end{pmatrix}}_{u} \longrightarrow L = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 2 & 5/4 & 1 \end{pmatrix}}_{L}$$

$$S^t = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \qquad S = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

$$S^t = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$S * S^t = A \iff \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 5 & 6 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 4 & 6 & 8 \\ 6 & 25 & 32 \\ 8 & 32 & 77 \end{pmatrix}}_{A}$$

Formules recurentes de calcule des coefficients $S$ :

- $j = 1$
- $i = 1 = j$ : $S_{11} = \sqrt{a_{11}}$ , $i > 1$, $j = 1$ : $S_{i1} = \dfrac{a_{i1}}{S_{11}}$
- $1 < (i = j) \leq n$ ou $2 \leq (i = j) \leq n$

$$S_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} S_{ik}^2} \equiv \text{ou} \quad S_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} S_{jk}^2}$$

- $j > 1$ et $i \neq j$

$$S_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} S_{ik} S_{jk}}{S_{jj}}$$

$i < j$ ou $j > i \iff S = 0$

ex : $S_{ij}$ : $S_{12} = 0$ , $S_{13} = 0$

$S_{23} = 0$

$\Delta_{11} = 4 = u_{11}$

$\Delta_2 = 64 = 4 \times 16 = u_{11} \times u_{22} \Rightarrow u_{22} = 16$

$\Delta_3 = 2304 = 4 \times 16 \times 36 = u_{11} \times u_{22} \times u_{33} = 4 \times 16 \times 36$

$= u_{11} \times u_{22} \times u_{33}$

**1.** $r = 1 \Leftrightarrow (j = 1)$ , $i = 1, 2, 3$

- $j = 1$ et $i = 1$ $\quad S_{11} = S_{ij} = S_{ii} = S_{jj}$ $\quad i = 1$ et $j = 1$

- $S_{11} = \sqrt{a_{11} - \sum_{k=1}^{0} \cancel{S_{10}^2}} = \sqrt{a_{11}} = \sqrt{4} = 2 \Leftrightarrow \boxed{S_{11} = 2}$

- $j = 1$ et $i > j$

$$S_{21} = \frac{a_{21} - \sum_{k=1}^{0} S_{20}\, \cancel{S_{10}}}{S_{11}} = \frac{a_{21}}{\sqrt{a_{11}}} = \frac{6}{2} = 3 \Leftrightarrow \boxed{S_{21} = 3}$$

$$S_{31} = \frac{a_{31} - \sum_{k=1}^{0} S_{30}\, \cancel{S_{10}}}{S_{11}} = \frac{a_{31}}{\sqrt{a_{11}}} = \frac{8}{2} = 4 \Leftrightarrow \boxed{S_{31} = 4}$$

**2.** $r = 2 \Leftrightarrow (j = 2)$

- $j = 2$ et $i = 2$

$$S_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} S_{2k}^2} = \sqrt{a_{22} - S_{21}^2} = \sqrt{25 - 3^2} = \sqrt{16} = 4 ; \boxed{S_{22} = 4}$$

$j - 1 = 2 - 1 = 1$

- $j = 2$ et $i = 3$

$$\underset{ij}{S_{32}} = S_{23} = \frac{a_{32} - \sum_{k=1}^{1} S_{3k}\, S_{2k}}{S_{22}} = \frac{a_{32} - S_{31} \cdot S_{21}}{S_{22}} = \frac{32 - 4 * 3}{4}$$

$$= \frac{20}{4} = 5 \Leftrightarrow \boxed{S_{32} = 5}$$

**3.** $r = 3$ $(j = 3)$

$j = 3$ et $i = 3$ $\quad j - 1 = 3 - 1 = 2$

$$S_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} S_{3k}^2} = \sqrt{a_{33} - (S_{31}^2 + S_{32}^2)} = \sqrt{77 - 16 - 25} = \sqrt{36}$$

$$\Leftrightarrow \boxed{S_{33} = 6}$$

$$Ax = B \Leftrightarrow S S^t . x = B \Leftrightarrow S \underbrace{S^t x}_{y} = B$$

$$\begin{cases} S . y = B \\ S^t . x = y \end{cases}$$

$$A = LU = LIU = \underbrace{L D^{1/2}}_{S} \underbrace{D^{-1/2} L^t}_{S}$$

$$S \underbrace{S^t x}_{y} = F$$

$$S Y = F \Leftrightarrow \begin{pmatrix} S_{11} & & 0 \\ S_{21} & S_{22} & \\ \vdots & & \ddots \\ S_{n1} & \cdots & S_{nn} \end{pmatrix} \begin{Bmatrix} y_1 \\ \vdots \\ y_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \vdots \\ F_n \end{Bmatrix}$$

$$\begin{cases} S_{11} y_1 = F_1 \\ S_{21} y_1 + S_{22} y_2 = F_2 \\ \vdots \\ S_{n1} y_1 + S_{n2} y_2 + \cdots S_{nn} y_n = F_n \end{cases} \Leftrightarrow \begin{cases} y_1 = \dfrac{F_1}{S_{11}} \\ y_2 = \dfrac{F_2 - (S_{21} y_1)}{S_{22}} \\ \vdots \\ y_n = \dfrac{F_n - (S_{n1} y_1 + S_{n2} y_2 + S_{n\,n-1} y_{n-1})}{S_{nn}} \end{cases}$$

$i = j$

$$y_i = \frac{F_i - \sum_{j=1}^{M} S_{ij} y_j}{S_{ii}}$$

matrice carré

$$y_i = \frac{F_n - (S_{n1} y_1 + S_{n2} y_2 + \cdots + S_{nn} y_n)}{S_{nn}}$$

$$\boxed{y_i = \frac{F_i - \sum_{j=1}^{M} S_{ij} y_j}{S_{ii}}}$$