Chapter Linear Programming Optimization Problem

1- Introduction

Linear programming is an optimization problem solving method applicable for solving problems in which both the objective function and the constraints appear as linear functions of the decision (design) variables.

The constraint equations in LP optimization problem may be under the form of equalities or inequalities.

LP is in fact a real revolutionary development that permits us to make optimal decisions in complex situations. Although it encompasses (comprises) several techniques, the Simplex method continues to be the most efficient and popular method for solving general LP optimization problems.

2- Standard form of Linear Programming Optimization Problem

The general linear programming optimization problem can be sated under the following standard form:

- **Scalar form**:

$$Minimize\ f(x_1, x_2, x_3, \ldots., x_n) = c_1x_1 + c_2x_2 + c_3x_3, \ldots., c_nx_n$$

$$subject\ to\ the\ constraints:$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \ldots., a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \ldots., a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \ldots., a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3, \ldots., a_{mn}x_n = b_m$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

$$\vdots$$

$$x_n \geq 0$$

Where:

$a_{ij}, c_j, b_j, (i = 1,2,3, \ldots., m,\ \ n = 1,2,3, \ldots \ldots, n)$; are known constants.

$x_j, (j = 1,2,3, \ldots., n)$: are known to be the design (decision) variables, which represent the solution to the optimization problem.

- **Matrix form**:

$$Minimize\ f(\boldsymbol{X}) = \boldsymbol{C}^T\boldsymbol{X}$$

$$subject\ to\ the\ constraints:$$

$$A\boldsymbol{X} = B$$

$$X \geq 0$$

Where:

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ . \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ . \\ b_m \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ . \\ m_n \end{bmatrix}$$

The characteristics of LP optimization problem as it is stated in the standard form are:

- The objective function is of the minimization type.
- All the constraints are of the equality type.
- All the decision variables are positive (nonnegative).

It is now shown that any linear programming optimization problem can be expressed in standard form by using the following transformations.

1. The maximization of the function $f(\boldsymbol{X})$ is equivalent the minimization of the negative of the same function, that is:

$$Maximize f(\boldsymbol{X}) \Leftrightarrow Minimize[-f(\boldsymbol{X})]$$

Consequently, the objective function can be stated in the minimization form in any linear programming problem.

2. If a constraint appears as in the form of inequality, that is "less than or equal to" such that:

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3, \dots, a_{kn}x_n \leq b_k$$

Then, it can be converted into equality type using the **slack variable**, that the conversion is performed by adding a **nonnegative** slack variable x_{n-1} as follows:

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3, \dots, a_{kn}x_n + x_{n-1} = b_k$$

3. Similarly, if a constraint is in the form of "greater than or equal to" type of inequality, that is:

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3, \dots, a_{kn}x_n \geq b_k$$

   It can be converted into equality type using the technique of **slack variable**. In this case, by subtracting the slack variable as:

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3, \dots, a_{kn}x_n - x_{n-1} = b_k$$

Where, in this case, the nonnegative variable x_{n-1} is called ***surplus variable***.

3- Solution of a System Linear Simultaneous Equations

Before studying the most general method of solving a linear programming optimization problem, it will be useful to review the methods of solving a system of linear equations.

Hence in the present section we review some of the elementary concepts of linear equations.

Consider the following system of n linear equations of n unknowns.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \dots, a_{1n}x_n = b_1 & (E_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \dots, a_{2n}x_n = b_2 & (E_2) \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3, \dots, a_{nn}x_n = b_n & (E_n) \end{cases} \Leftrightarrow A\boldsymbol{X} = B \qquad (1)$$

Assuming that this set of equations simultaneously possesses a unique solution.

A method of solving this system of equations consists of reducing the equations to a form known as canonical form. It is well known from elementary algebra that the solution to the system (1) will not be altered the following elementary operations:

- ✓ Any equation E_r is replaced by the equation kE_r, where k is a nonzero constant.
- ✓ Any equation E_r is replaced by the equation $E_r + kE_s$, where E_s, is any other equation of the system.

By making use of these elementary operations, the system of equations (1) can be reduced to a convenient equivalent form as follows.

Let us select some variable x_i and try to eliminate it from all the equations except the jth one (for which $a_{ji} \neq 0$). This can be accomplished by dividing the jth equation by a_{ji} and subtracting a_{ki} times the results from each of the other equations, $k = 1,2,3, \ldots., (j-1), (j+1), \ldots \ldots, n$.

The resulting system of equation can be written as:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3, \ldots. +0x_n = b_1'' \\ 0x_1 + 1x_2 + 0x_3, \ldots. +0x_n = b_2'' \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ 0x_1 + 0x_2 + 0x_3, \ldots. +1x_n = b_n'' \end{cases} \quad (2)$$

The procedure of eliminating a particular variable from all but one equation is called a ***<u>pivot operation</u>***.

The system of equations obtained by the **<u>pivot operation</u>** has exactly the same solution as the original set of equations (1). This is the vector $\boldsymbol{X}$ that satisfies both sets (1) and (2) and vice versa.

Next, if we take the system of equations (2) and perform a new ***pivot operation*** by eliminating $x_s, for\ s \neq i$, in all the equations except the $t\ th$ equation, $t \neq j$, the zeros or the 1s in the ith column will not be disturbed. The ***pivotal operations*** can be repeated by using a different variable and equation each time until the system of Eqs. (1) is reduced to the form:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3, \ldots. +0x_n = b_1'' \\ 0x_1 + 1x_2 + 0x_3, \ldots. +0x_n = b_2'' \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ 0x_1 + 0x_2 + 0x_3, \ldots. +1x_n = b_n'' \end{cases} \quad (3)$$

This system of equations (3) is said to be ***in canonical form*** and has been obtained after carrying out n pivot operations. From the ***canonical form***, the solution vector can be directly obtained as:

$$x_i = b_i'', \quad 1,2,3, \ldots., n \quad\quad (4)$$

Since the set of equations (3) has been obtained from equations (1) only through elementary operations, the system of equations (3) is equivalent to the system of equations (1). Thus the solution given by equation (4) is the desired solution of the set of equations (1).

4- Pivotal Reduction of a General System of Equations

Instead of a **square** system, let us consider a system of m equations in n variables with $n \geq m$. This system of equations is assumed to be consistent so that it will have at least one solution:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \ldots., a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \ldots., a_{2n}x_n = b_2 \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3, \ldots., a_{mn}x_n = b_m \end{cases} \quad (5)$$

The solution vector(s) $\boldsymbol{X}$ that satisfy the set of equations (5) are not evident from the equations. However, it is possible to reduce this system to an equivalent ***canonical system*** from which at least one solution can readily be deduced. If pivotal operations with respect to any set of m variables, say, $x_1, x_2, x_3, \ldots., x_m$, are carried out, the resulting set of equations can be written as follows:

One special solution that can always be deduced from the system of equations (5) is

$$x_i = \begin{cases} b_i'' , & i = 1,2,3, \ldots , m \\ 0, & i = m+1, m+2, m+3, \ldots , n \end{cases} \tag{7}$$

This solution is called a ***basic solution*** since the solution vector contains no more than m nonzero terms. The pivotal variables x_i, $i = 1, 2, \ldots, m$, are called the ***basic variables*** and the other variables x_i, $i = m + 1, m + 2, \ldots, n$, are called the ***nonbasic variables***.

Of course, this is not the only solution, but it is the one most readily deduced from equations (6). If all b_i'', $i = 1,2,3, \ldots, m$, in the solution given by Eqs. (7) are nonnegative, it satisfies Eqs. (???) in addition to Eqs. (????),defining the LP optimization problem, and hence it can be called a ***basic feasible solution***.

It is possible to obtain the other basic solutions from the ***canonical system*** of Eqs. (6). We can perform an additional pivotal operation on the system after it is in canonical form, by choosing a''_{pg} (which is nonzero) as the pivot term, $q > m$, and using any row p (among $1, 2, \ldots, m$). The new system will still be in canonical form but with x_q as the pivotal variable in place of x_p. The variable x_p, which was a basic variable in the ***original canonical form***, will no longer be a basic variable in the ***new canonical form***. This new canonical system yields a new basic solution (which may or may not be **feasible**) similar to that of Eqs. (7). It is to be noted that the values of all the basic variables change, in general, as we go from one **basic solution** to another, but only one zero variable (which is nonbasic in the original canonical form) becomes nonzero (which is basic in the new canonical system), and vice versa.

Example:

Find the basic solution corresponding to the system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 & (I_0) \\ x_1 + x_2 + x_3 + 3x_4 = 6 & (II_0) \\ x_1 - x_2 + x_3 + 5x_4 = 4 & (III_0) \end{cases}$$

Solution:

First we reduce the system of equations into canonical form with x_1, x_2, x_3 as basic variables. To do that, we pivot on the element $a_{11} = 2$ to obtain:

$$\begin{cases} x_1 + \frac{3}{2}x_2 - x_3 - \frac{7}{2}x_4 = \frac{1}{2} & I_1 = \frac{I_0}{2} \\ 0x_1 - \frac{1}{2}x_2 + 2x_3 + \frac{13}{2}x_4 = \frac{11}{2} & II_1 = II_0 - I_1 \\ 0x_1 - \frac{5}{2}x_2 + 2x_3 + \frac{17}{2}x_4 = \frac{7}{2} & III_1 = III_0 - I_1 \end{cases}$$

Now, we pivot on $a_{22}{}' = -\frac{1}{2}$, we get:

$$\begin{cases} x_1 + 0x_2 + 5x_3 + 16x_4 = 17 & I_2 = I_1 - \frac{3}{2}II_2 \\ 0x_1 + x_2 - 4x_3 - 13x_4 = -11 & II_2 = -2II_1 \\ 0x_1 + 0x_2 - 8x_3 - 24x_4 = -24 & III_2 = III_1 + \frac{5}{2}II_2 \end{cases}$$

Now, we pivot on $a_{33}{}' = -8$, we get:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 + x_4 = +2 & I_3 = I_2 - 5III_3 \\ 0x_1 + 1x_2 + 0x_3 - x_4 = +1 & II_3 = II_2 + 4III_3 \\ 0x_1 + 0x_2 + 1x_3 + 3x_4 = +3 & III_3 = -\frac{1}{8}III_2 \end{cases}$$

From this canonical form, we can readily write the basic solution consisting in the basic variables (x_1, x_2, x_3) in terms of the variable x_4 as:

$$\begin{cases} x_1 = 2 - x_4 \\ x_2 = 1 + x_4 \\ x_3 = 3 - 3x_4 \end{cases}$$

If Eqs. (I0), (II0), and (III0) are the constraints of a linear programming optimization problem, the solution obtained by setting the ***independent variable*** x_4 equal to zero is called a **basic solution**. In the present case, the basic solution is given by:

$$\begin{cases} x_1 = 2 - 0 \\ x_2 = 1 + 0 \\ x_3 = 3 - 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = 3 \end{cases}$$

And the variables (x_1, x_2, x_3) are called basic variables, and x_4 is called nonbasic (or independent) variable.

Since this basic solution has all $x_j \geq 0$ $(j = 1,2,3,4)$, it is a **basic feasible solution**.

If we want to move to a **neighboring** of the basic solution, we can proceed from the canonical form given by Eqs. (I3), (II3), and (III3). Thus if a **canonical form** in terms of the variables (x_1, x_2, x_4) is required, we have to bring x_4 into the ***basis*** in place of the original basic variable x_3. Hence we pivot on $a_{34}{}'' = 3$ in Eq. (III3). This gives the desired canonical form as :

$$\begin{cases} 1x_1 + 0x_2 + 0x_4 - \frac{1}{3}x_3 = 1 & I_4 = I_3 - III_4 \\ 0x_1 + 1x_2 + 0x_4 + \frac{1}{3}x_3 = 2 & II_4 = II_3 + III_4 \\ 0x_1 + 0x_2 + 1x_4 + \frac{1}{3}x_3 = 1 & III_4 = +\frac{1}{3}III_3 \end{cases}$$

This new canonical system gives the new basic solution of (x_1, x_2, x_4) in terms of x_3 as

$$\begin{cases} x_1 = 1 + \frac{1}{3}x_3 \\ x_2 = 2 - \frac{1}{3}x_3 \\ x_4 = 1 - \frac{1}{3}x_3 \end{cases}$$

And the corresponding basic solution (x_1, x_2, x_4) is obtained by taking the nonbasic (independent) variable equals zero; $x_3 = 0$ as:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_4 = 1 \\ x_3 = 0 \end{cases}$$

This basic solution can also be seen to be a basic feasible solution.

If we want to move to the next basic solution with (x_1, x_3, x_4) as basic variables, we have to bring x_3 into the current basis in place of x_2. Thus we have to pivot on $a_{24}'' = \frac{1}{3}$ in Eq. (II4). This leads to the following canonical system:

$$\begin{cases} 1x_1 + 0x_3 + 0x_4 + 1x_2 = 3 & I_5 = I_4 + \frac{1}{3}II_5 \\ 0x_1 + 1x_3 + 0x_4 + 3x_2 = 6 & II_5 = 3II_4 \\ 0x_1 + 0x_3 + 1x_4 - 1x_2 = -1 & III_5 = III_4 - \frac{1}{3}II_5 \end{cases}$$

This new canonical system gives the new basic solution of (x_1, x_3, x_4) in terms of x_2 as:

$$\begin{cases} x_1 = 3 - x_2 \\ x_3 = 6 - 3x_2 \\ x_4 = -1 + x_2 \end{cases}$$

And the corresponding basic solution (x_1, x_3, x_4) is obtained by taking the nonbasic (independent) variable equals zero; $x_2 = 0$ as:

$$\begin{cases} x_1 = 3 \\ x_3 = 6 \\ x_4 = -1 \\ x_2 = 0 \end{cases}$$

Since all the x_j are **not nonnegative**, this **basic solution** is **not feasible**.

Finally, to obtain the canonical form in terms of the basic variables (x_2, x_3, x_4), we pivot on $a_{12}'' = 1$ in Eq. (I5), thereby bringing x_2 into the current basis in place of x_1. This gives:

$$\begin{cases} 1x_2 + 0x_3 + 0x_4 + 1x_1 = 3 & I_6 = I_5 \\ 0x_2 + 1x_3 + 0x_4 - 3x_1 = -3 & II_6 = II_5 - 3I_6 \\ 0x_2 + 0x_3 + 1x_4 + 1x_1 = 2 & III_6 = III_5 + I_6 \end{cases}$$

This new canonical system gives the new basic solution of (x_2, x_3, x_4) in terms of x_1 as:

$$\begin{cases} x_2 = 3 - x_1 \\ x_3 = -3 + 3x_1 \\ x_4 = 2 - x_1 \end{cases}$$

And the corresponding basic solution (x_2, x_3, x_4) is obtained by taking the nonbasic (independent) variable equals zero; $x_1 = 0$ as:

$$\begin{cases} x_2 = 3 \\ x_3 = -3 \\ x_4 = 2 \\ x_1 = 0 \end{cases}$$

Since all the x_j are **not nonnegative**, this **basic solution** is **not feasible**.

5- Motivation to the Simplex Method

Given a system in canonical form corresponding to a basic solution, we have seen how to move to a neighboring basic solution by a pivot operation. Thus one way to find the optimal solution of the given linear programming problem is to generate all the basic solutions and pick the one that is feasible and corresponds to the optimal value of the objective function. This can be done because the optimal solution, if one exists, always occurs at an extreme point or vertex of the feasible domain. If there are m equality constraints in n variables with $n \geq m$, a basic solution can be obtained by setting any of the $(n - m)$ variables equal to zero. The number of ***basic solutions*** to be inspected is thus ***equal*** to the number of ways in which m variables can be selected from a set of n variables, that is,

$$\binom{n}{m} = \frac{n!}{(n - m)!\, m!}$$

Example:

if $n = 10$ and $m = 5$, we have 252 basic solutions, and if $n = 20$ and $m = 10$, we have 184,756 basic solutions.

Usually, we do not have to inspect all these basic solutions since many of them will be ***infeasible***. However, for large values of n and m, this is still a very large number to inspect one by one. Hence what we really need is a computational ***scheme*** that examines a sequence of ***basic feasible solutions***, each of which corresponds to a lower value of f until a ***minimum*** is reached. The ***Simplex*** method of Dantzig is a ***powerful scheme*** for obtaining a ***basic feasible solution***; if the solution is ***not optimal***, the method provides for finding a ***neighboring*** ***basic feasible solution*** that has a lower or equal value of f . The process is ***repeated*** until, in a finite number of steps, an **optimum** is found.

The first step involved in the ***Simplex*** method is to construct an ***auxiliary problem*** by introducing certain variables known as ***artificial variables*** into the ***standard form of the linear programming problem***. The primary aim of adding the ***artificial variables*** is to bring the resulting ***auxiliary problem*** into a ***canonical form*** from which its ***basic feasible solution*** can be obtained immediately. Starting from this canonical form, the **optimal solution** of the original linear programming problem is sought in two phases. The first phase is intended to find a ***basic feasible solution*** to the original linear programming problem. It consists of a sequence of ***pivot operations*** that produces a succession of different ***canonical forms*** from which the **optimal solution** of the ***auxiliary problem*** can be found. This also enables us to find a **basic feasible solution**, if one exists, of the original linear programming problem. The second phase is intended to find the **optimal solution** of the original linear programming problem. It consists of a second sequence of **pivot**

operations that enables us to ***move*** from one ***basic feasible solution*** to the next of the original linear programming problem. In this process, the optimal solution of the problem, if one exists, will be identified.

The sequence of different canonical forms that is necessary in both the phases of the ***Simplex*** method is generated according to the ***Simplex algorithm*** described in the next section. That is, the simplex algorithm forms the main ***subroutine*** of the ***Simplex*** method.

6- Simplex Algorithm

The starting point of the simplex algorithm is always a set of equations, which includes the objective function along with the equality constraints of the problem in canonical form. Thus the objective of the simplex algorithm is to find the vector $\boldsymbol{X} \geq 0$ that minimizes the function $f(\boldsymbol{X})$ and satisfies the equations:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 + \cdots . +0x_m = b_1'' \\ 0x_1 + 1x_2 + 0x_3 + \cdots . +0x_m = b_2'' \\ \vdots \\ 0x_1 + 0x_2 + 0x_3 + \cdots . +1x_m = b_n'' \\ 0x_1 + 0x_2 + 0x_3 + \cdots . +0x_m - f + c''_{m+1}x_{m+1} + \cdots + c''_{mn}x_n = -f''_0 \end{cases} \qquad (10)$$

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If the basic solution is also feasible, the values of x_i , $i = 1, 2, \ldots, n$, are nonnegative and hence:

$$b_i'' \geq 0, \quad i = 1, 2, \ldots, m,$$

In **phase I** of the Simplex method, the basic solution corresponding to the **canonical form** obtained after the introduction of the **artificial variables** will be **feasible** for the ***auxiliary problem***. As stated earlier, ***phase II*** of the Simplex method starts with a ***basic feasible solution*** of the ***original*** linear programming problem. Hence the initial canonical form at the start of the simplex algorithm will always be a ***basic feasible solution***.

We know from ***Theorem 3.6*** that the ***optimal solution*** of a linear programming problem ***lies at one of*** the ***basic feasible solutions***. Since the simplex algorithm is intended to move from one basic feasible solution to the other through pivotal operations, before moving to the next basic feasible solution, we have to make sure that the present basic feasible solution is not the optimal solution. By merely ***glancin***g at the numbers c''_j , $j = 1, 2, \ldots, n$, we can tell whether or not the present basic feasible solution is optimal. ***Theorem 3.7*** provides a means of identifying the optimal point.

7- Identifying an Optimal Point (Solution)

Theorem 3.7:

A basic feasible solution is an ***optimal solution*** with a minimum objective function value of f''_0 if all the cost coefficients c''_j , $j = m + 1, m + 2, \ldots, n$, in Eqs. (10) are nonnegative.

Illustrative Example: Simplex Method to solve LP problem

Consider the LP optimization problem, which state as follows:

$$Maximize\ f(\boldsymbol{X}) = x_1 + 2x_2 + x_3$$

$$subject\ to\ the\ constraints:$$

$$\begin{cases} 2x_1 + x_2 - x_3 \leq 2 \\ -2x_1 + x_2 - 5x_3 \geq -6 \\ 4x_1 + x_2 + x_3 \leq 6 \end{cases} \quad and: \quad x_i \geq 0,\ \ i = 1,2,3$$

Solution:

Step 1: we first change the sign of the objective function to convert it to a minimization problem and the signs of the inequalities (where necessary) so as to obtain nonnegative values of the constants b_i (to see whether an initial basic feasible solution can be readily obtained).

By doing this, the resulting optimization problem can be stated (formulated) as follows:

$$Minimize\ [-f(\boldsymbol{X})] = -x_1 - 2x_2 - x_3$$

$$subject\ to\ the\ constraints:$$

$$\begin{cases} 2x_1 + x_2 - x_3 \leq 2 \\ 2x_1 - x_2 + 5x_3 \leq 6 \\ 4x_1 + x_2 + x_3 \leq 6 \end{cases} \quad and: \quad x_i \geq 0,\ \ i = 1,2,3$$

To convert the inequality constraints to equality constraints we add respectively the nonnegative slack variables $x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0$. Hence, the system of equations can be stated in the canonical form as:

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 2 \\ 2x_1 - x_2 + 5x_3 + x_5 = 6 \\ 4x_1 + x_2 + x_3 + x_6 = 6 \\ -x_1 - 2x_2 - x_3 - f = 0 \end{cases} \quad and: \quad x_i \geq 0,\ \ i = 1,2,3,4,5,6 \qquad (E_1)$$

Where $x_4, x_5, x_6\ \ and - f$ can be considered as basic variables.

The set of equations (E_1) can be put in a canonical form as follows:

$$\begin{cases} 2x_1 + x_2 - x_3 + 1x_4 + 0x_5 + 0x_6 = 2 \\ 2x_1 - x_2 + 5x_3 + 0x_4 + 1x_5 + 0x_6 = 6 \\ 4x_1 + x_2 + x_3 + 0x_4 + 0x_5 + 1x_6 = 6 \\ -x_1 - 2x_2 - x_3 - f = 0 \end{cases} \qquad (E_1)$$

The basic solution corresponding to the basic variables (x_4, x_5, x_6) is obtained by letting the nonbasic variables equals zero, that is: $x_1 = x_2 = x_3 = 0$.

The basic solution is:

$$\begin{cases} x_4 = 2 \\ x_5 = 6 \\ x_6 = 6 \\ f = 0 \end{cases} \qquad (E_2)$$

Since: $b_i \geq 0, i = 1,2,3$, the obtained basic solution is feasible.

Question: **is this basic feasible solution optimal**?

To answer this question, we apply the theorem 3.7.

We note that the cost coefficients are not nonnegative ($c_i'' < 0, i = 1,2,3$) (that is: $c_1'' = -1, c_2'' = -2, c_1'' = -1$), the present basic feasible solution in not optimal.

Therefore, we think to improve this solution by first deciding the variable x_s to be brought to the basis. The nonbasic variable x_s to be brought to the basis is the one satisfying the following:

$$c_s'' = \min\ (c_j'' < 0)$$

By looking at the nonbasic variables and their corresponding coefficients, we find:

$$c_s'' = \min\left(c_j'' < 0\right) = \min(-1, -2, -1) = -2 = c_2''$$

That is, the nonbasic variable x_2 is going to be brought to the basis. This is done by pivotal operation on the pivot element to obtain the new canonical form of the system.

To obtain the new canonical form, we select the pivot element such that:

$$\frac{b_r''}{a_{rs}''} = \min_{a_{is}''>0} \left(\frac{b_i''}{a_{is}''}\right)$$

In this case:

$$s = 2, \qquad a_{12}'', a_{32}'' > 0, a_{22}'' < 0$$

Since:

$$\frac{b_1''}{a_{12}''} = \frac{2}{1} = 2$$

$$\frac{b_3''}{a_{32}''} = \frac{6}{1} = 6$$

Hence, the pivot element is: a_{12}''

By pivoting on a_{12}'', the new system of equations can be obtained as:

$$\begin{cases} 2x_1 + 1x_2 - x_3 + x_4 + 0x_5 + 0x_6 = 2 & I_1 = I_0 \\ 4x_1 + 0x_2 + 4x_3 + 1x_4 + 1x_5 + 0x_6 = 8 & II_1 = II_0 + I_1 \\ 2x_1 + 0x_2 + 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4 & III_1 = III_0 - I_1 \\ 3x_1 + 0x_2 - 3x_3 + 2x_4 \qquad\qquad - f = 4 & IIII_1 = IIII_0 + 2I_1 \end{cases}$$

Therefore, the basic variables are (x_2, x_5, x_6) and the nonbasic variables are: (x_1, x_3, x_4).

The basic solution is obtained by making the nonbasic variables equal zero, that is:

$$\begin{cases} x_2 = 2 \\ x_5 = 8 \\ x_6 = 4 \\ f = -4 \end{cases} \qquad (E_3)$$

Is this solution feasible? Yes, it is.

Is this solution optimum?

According to Theorem 3.7, we have: $c_3'' = -3 < 0\ (negative)$, hence the obtained solution is not optimum.

Therefore, we proceed to improve the present solution using pivotal operation.

- We decide which nonbasic variable to enter the basis, this corresponds to

$$c_s'' = \min(c_j'' < 0) = \min(-3) = -3 = c_3''$$

- We decide which element is the pivot element, this satisfies:

$$\frac{b_r''}{a_{rs}''} = \min_{a_{is}''>0}\left(\frac{b_i''}{a_{is}''}\right) = \min\left(\frac{8}{4}, \frac{4}{2}\right) = \min\,(2,2)$$

We notice that the two ratios are the same, then we arbitrary select one, say: $\frac{8}{4}$,

Then the pivot element is; a_{33}''. By pivoting on this element, the new canonical form of the system of equations will be as follows:

$$\begin{cases} 3x_1 + 1x_2 + 0x_3 + \frac{5}{4}x_4 + \frac{1}{4}x_5 + 0x_6 = 4 & I_2 = I_1 + II_2 \\ x_1 + 0x_2 + 1x_3 + \frac{1}{4}x_4 + \frac{1}{4}x_5 + 0x_6 = 2 & II_2 = \frac{1}{4}II_1 \\ 0x_1 + 0x_2 + 0x_3 - \frac{1}{2}x_4 - \frac{1}{2}x_5 + 1x_6 = 0 & III_2 = III_1 - 2II_2 \\ 6x_1 + 0x_2 + 0x_3 + \frac{11}{4}x_4 + \frac{3}{4}x_5 \qquad - f = 10 & IIII_2 = IIII_1 + 3II_2 \end{cases}$$

Corresponding to this canonical form, the basic solution is:

$$\begin{cases} x_2 = 4 \\ x_3 = 2 \\ x_6 = 0 \\ f = -10 \end{cases} \qquad (E_5)$$

In this solution, all $b_i'' \geq 0$, that is it is feasible.

Is it optimum?

All c_j'' *are nonnegative*, hence, the obtained solution is optimum with a minimum value of the objective function equals (- 10).

Usually, starting from step (E_1) to the final step (E_5), all the computations are done in a tableau form as shown below:

Basic variables	Variables							b_i''	$\frac{b_i''}{a_{is}''}, a_{is}'' > 0$
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$		
$\boldsymbol{x_4}$	2	$\boxed{1}$	-1	1	0	0	0	2	2← (*the smaller one*)
$\boldsymbol{x_5}$	2	-1	5	0	1	0	0	6	not taken
$\boldsymbol{x_6}$	4	1	1	0	0	1	0	6	6

$-f$	-1	-2 $\uparrow$	-1	0	0	0	1	0	

The most negative cost coefficient $c_i^{''} = c_2^{''}$ (indicating that x_2 is to enter the next basis)

Result of pivoting:

Basic variables	**Variables**							$b_i^{''}$	$\frac{b_i^{''}}{a_{is}^{''}}, a_{is}^{''} > 0$
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$		
x_2	2	1	-1	1	0	0	0	2	Not taken
x_5	4	0	$\boxed{4}$	1	1	0	0	8	2← *(the smaller one)*
x_6	2	0	2	-1	0	1	0	4	2
$-f$	3	0	-3 $\uparrow$	2	0	0	1	4	

The most negative cost coefficient $c_i^{''} = c_3^{''}$ (indicating that x_3 is to enter the next basis).

The results of pivoting is:

Basic variables	**Variables**							$b_i^{''}$	$\frac{b_i^{''}}{a_{is}^{''}}, a_{is}^{''} > 0$
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$		
x_2	3	1	0	5/4	1/4	0	0	4	
x_3	1	0	$\boxed{1}$	1/4	1/4	0	0	2	
x_6	0	0	0	-3/2	-1/2	1	0	0	
$-f$	6	0	0	11/4	3/4	0	1	10	

All $c_i^{''} \geq 0$, hence, the present solution is optimum.

And the minimum value of the objective function is (- 10).

8- Two Phases of Simplex Method

The LP optimization problem is stated as to find the nonnegative values for the design (decision) variables: $(x_1, x_2, x_3, \ldots, x_n)$ that satisfy the system of constraint equations and minimize the objective function. That is the intention is to solve the optimization problem:

$$Minimize\ f(x_1, x_2, x_3, \ldots., x_n) = c_1x_1 + c_2x_2 + c_3x_3, \ldots., c_nx_n \tag{11}$$

$$subject\ to\ the\ constraints:$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \ldots., a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \ldots., a_{2n}x_n = b_2 \\ \vdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3, \ldots., a_{mn}x_n = b_m \end{cases} \tag{12}$$

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$
$$.$$

$$.$$
$$.$$
$$x_n \geq 0$$

The general problems encountered when solving this problem are:

(1) An initial feasible canonical may not be readily be available, which is the case when the LP problem does not have slack variables for some of the equations or when the slack variables have ***negative coefficients***.

(2) The optimization problem may have redundancies and / or inconsistencies, and may not be solvable in nonnegative numbers.

The two-phase simplex method can be used to solve the problem.

Phase I of the Simplex method uses the Simplex algorithm itself to find whether the LP optimization problem has a feasible solution. If a feasible solution exists, it provides a basic feasible solution in canonical form ready to initiate phase II of the method.

Phase II, in turn, uses the Simplex algorithm to find whether the problem has a bounded optimum solution. If this bounded optimum exists, it finds the basic feasible solution that is optimum.

The Simplex method is described in the following steps.

Step 1: arrange the original form of the Eqs. (12) so that all constant terms $b_i, i = 1,2, \dots, m$ are positive or zero by changing, where necessary, the signs on both sides of any of the equations.

Step 2: introduce to this system of equations a set of artificial variables (called slack variables) $y_i, i = 1,2, \dots, m$, which serve as basic variables in phase I, where: $y_i \geq 0$, so that the system of equations becomes:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \dots, a_{1n}x_n + y_1 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \dots, a_{2n}x_n + y_2 = b_2 \\ \vdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3, \dots, a_{mn}x_n + y_m = b_m \\ y_i \geq 0 \end{cases} \tag{13}$$

Step 3: arrange the objective function under the form:

$$c_1x_1 + c_2x_2 + c_3x_3, \dots, c_nx_n - f = 0 \tag{14}$$

Step 4: Phase I:

Step 5:

Step 6: Phase II of the method

Example:

$$Minimize\ f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$$
$$subject\ to\ the\ constraints:$$
$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 = 0$$
$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 2$$
$$x_i \geq 0, i = 1,2, \dots ,5$$

Solution:

Step 1: since the constants on the right-hand side of the constraint equations are already nonnegative, the application of step 1 is unnecessary.

Step 2: we arrange the objective function to have the form:

$$2x_1 + 3x_2 + 2x_3 - x_4 + x_5 - f = 0$$

Step 3: we introduce the artificial variables $y_i \geq 0, i = 1,2$.

As a result, the equations formulating the optimization problem become:

$$\begin{cases} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 & = 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 + \qquad y_2 & = 2 \\ 2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad - f & = 0 \end{cases}$$

Step 4: we define the infeasibility equation as the sum of the artificial variables as:

$$\sum_{i=1}^{m} y_i = w$$

In this case, we get:

$$\sum_{i=1}^{2} y_i = w \Rightarrow y_1 + y_2 = w$$

Consequently, the complete array of equations can be written as:

$$\begin{cases} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 & = 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 & = 2 \\ 2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad - f & = 0 \\ \qquad y_1 + y_2 \qquad - w & = 0 \end{cases} \qquad (E_2)$$

Obviously, this array can be rewritten as a canonical form with basic variables $(y_1, y_2, -f, -w)$ by subtracting the sum of first two equations of (E_2) from the last equation of (E_2).

Thus the last equation of (E_2) becomes:

$$-4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 + 0y_1 + 0\,y_2 - w \;= -2 \qquad (E_3)$$

We can now reformulate the LP optimization problem by the following system of equations:

$$\begin{cases} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 = 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 = 2 \\ 2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad - f = 0 \\ -4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 + 0y_1 + 0\,y_2 - w \;= -2 \end{cases} \qquad (E_4)$$

Since this canonical system [first three equations of (E_2) and the equation (E_3)] provides an initial basic feasible solution, phase I of Simplex method can be started.

Phase I computations are shown below in tableau form:

Basic variables	Variables									b_i''	$\frac{b_i''}{a_{is}''}, a_{is}'' > 0$
	x_1	x_2	x_3	x_4	x_5	y_1	y_2	$-f$	$-w$		
$\boldsymbol{y_1}$	3	-3	4	$\boxed{2}$	-1	1	0	0	0	0	0
$\boldsymbol{y_2}$	1	1	1	3	1	0	1	0	0	2	2/3
$\boldsymbol{-f}$	2	3	2	-1	1	0	0	1	0	0	
$\boldsymbol{-w}$	-4	**2**	-5 ↑	-5 ↑	0	0	0	0	1	-2	

Most negative coefficients

Since there is a tie between two most negative coefficients d_3'' and d_4'', we take arbitrarily d_4'' as the most negative d_i'' for choosing which nonbasic variable to enter the basis and which one to drop from the basis. In this case, x_4 is being enter the basis.

The result of pivoting is shown in the following tableau.

Basic variables	Variables									b_i''	$\frac{b_i''}{a_{is}''}, a_{is}'' > 0$
	x_1	x_2	x_3	x_4	x_5	y_1	y_2	$-f$	$-w$		
$\boldsymbol{x_4}$	3/2	-3/2	2	$\boxed{1}$	-1/2	1/2	0	0	0	0	
$\boldsymbol{y_2}$	-7/2	$\boxed{\frac{11}{2}}$	-5	0	5/2	-3/2	1	0	0	2	4/11
$\boldsymbol{-f}$	7/2	3/2	4	0	1/2	1/2	0	1	0	0	
$\boldsymbol{-w}$	-2/3	**-11/2** ↑	5	0	-5/2	5/2	0	0	1	-2	

Most negative

The result of pivoting on $a_{is}'' = a_{22}''$ is shown in the following tableau.

Basic variables	Variables									b_i''	$\frac{b_i''}{a_{is}''}$, $a_{is}'' > 0$
	x_1	x_2	x_3	x_4	x_5	y_1	y_2	$-f$	$-w$		
$\boldsymbol{x_4}$	6/11	0	6/11	1	2/11	1/11	3/11	0	0	0	
$\boldsymbol{x_2}$	-7/11	$\boxed{1}$	-10/11	0	5/11	-3/11	2/11	0	0	3/22	
$\boldsymbol{-f}$	49/11	0	59/11	0	2/11	10/11	-3/11	1	0	-9/44	
$\boldsymbol{-w}$	0	**0**	0	0	0	0	0	0	0	0	

Step 5: at this stage, we notice that the present basic feasible solution does not contain any of the artificial variables and also the value of w is reduced to zero ($w = 0$) indicating that phase I is completed.

Step 6: Phase II:

Now we start phase II computations by dropping the w row from further consideration. The result of phase II is again shown in tableau form as:

Basic variables	Variables									b_i''	$\frac{b_i''}{a_{is}''}$, $a_{is}'' > 0$
	x_1	x_2	x_3	x_4	x_5			$-f$			
$\boldsymbol{x_4}$	6/11	0	6/11	1	2/11			0		0	
$\boldsymbol{x_2}$	-7/11	$\boxed{1}$	-10/11	0	5/11			0		3/22	
$\boldsymbol{-f}$	49/11	0	59/11	0	2/11			1		-9/44	

At the end, since all the cost coefficients c_i'' are nonnegative, phase II is completed and the unique optimal solution is given by:

$$x_1 = x_2 = x_3 = 0 \qquad (nonbasic\ variables)$$

$$x_4 = \frac{2}{5}, \quad x_5 = \frac{4}{5} \qquad (basic\ variables)$$

$$f_{min} = \frac{2}{5}$$