Chapter 4: Performance Analysis of discrete linear feedback control systems

The main objective of designing any feedback control system is to ensure its stability in general conditions but also this stability is required to be robust against some environmental properties and influences. This chapter will be fundamentally devoted to tackle in study and analysis the main performance properties of discrete and linear feedback control systems where the behavior of each system is represented by the corresponding discrete (or Z) transfer function. Mainly we will focus on how to study the stability and accuracy performance indices.

1. Stability of discrete control systems

In general, the stability of any closed loop control system is related to the value of the amplitude of the steady state response. In other words, the feedback control system can be viewed stable if the amplitude of its output response has a finite value as time tends to infinity. Conversely, the response amplitude value is unbounded, the control system is unstable.

In this chapter our interest will be focused on defining and discussing the stability of linear time invariant discrete systems from two different perspectives; that is the stability regarding the steady state magnitude of the system response, where the asymptotic and marginal stability are to be defined. Similarly, the stability regarding the input and output characteristics which is called the stability in the sense of bounded input and bounded output (BIBO) will also be defined. Whereas the second perspective will be that of the stability based on pole locations of the corresponding discrete transfer function used to describe the behavior of the system.

1.1. Asymptotic and Marginal Stability

The definition of asymptotic and marginal stability is mainly related to the way and manner the system's output response responds to the initial conditions. If it decays asymptotically to zero corresponding to the applied initial conditions, then we are talking about the asymptotic stability. We mathematically express this as:

$$\lim_{k\to\infty} y(k) = 0 \tag{4.1}$$

Where:

$y(k)$: is the output response of the discrete system.

If however, the system's response does not decay to zero due to the applied initial condition and stays bounded, in this case we are talking about marginal stability. We can mathematically express this as:

$$\lim_{k\to\infty} y(k) = C\,,\;\; C \neq 0 \tag{4.2}$$

Where:

C: is any constant.

1.2. Bounded Input-Bounded Output Stability

The bounded input bounded output stability is defined based on the forced response with respect to the applied bounded input. To understand this definition, we introduce and define mathematically the bounded discrete input as follows:

if $u(k)$ be a discrete signal with: $k = 0,1,2,3, \dots \dots , \infty$. then $u(k)$ is said to be bounded if its samples are upper limited; in other words, there exists a real number $M = B_u > 0$, such that:

$$\forall\, k \in \mathbb{N}, |u(k)| < M \tag{4.3}$$

Consequently and based on the steady state amplitude or the magnitude of the output response, we can mathematically define the bounded input, bounded output stability as:

$$|u(k)| < b_u, \;\; \begin{cases} k = 0,1,2,, \dots \dots \\ 0 < b_u < \infty \end{cases} \Rightarrow |y(k)| < b_y, \;\; \begin{cases} k = 0,1,2,, \dots \dots \\ 0 < b_y < \infty \end{cases} \tag{4.4}$$

With:

$u(k), y(k)$: Are respectively the input and output of the discrete system.

b_u, b_y: Are respectively the input and output upper bounds.

1.3. Stability Definition in the Complex Z plane

1.3.1. Mapping between s and z complex planes

What is more important regarding the stability analysis and study of any linear time invariant closed loop control system whatever whether it is continuous time or discrete time (or sampled data) is the absolute and relative stability which are also known respectively as asymptotic and marginal stability. These two types of stability are hopefully determined by observing the location of the poles of the system's closed loop transfer function in a complex plane formed by the two perpendicular axes called respectively as real axis and imaginary axis.

Concerning the discrete feedback control system stability analysis, a mapping between the continuous time complex plane (s-plane) and the discrete time complex plane (z-plane) is generally established. As it is known, a continuous time closed loop control system stability is studied and analyzed according to the location of its poles in the complex s-plane in such a way the system is said to be absolutely stable when all its poles are located on the left hand side (LHS) of the s-plane, but if at least one pole is found on the Right Hand Side (RHS) of the s-plane, the system then is said to be unstable. Analogously, we can apply the same approach to determine the stability state of the discrete time control system but in the z-plane which is jointly related to the s-plane according to the following demonstration.

We have that the complex variables s and z are related by:

$$z = e^{sT_s} \tag{4.5}$$

With:

T_s: is the sapling period.

s: is the Laplace complex variable.

If we consider that:

$$s = \sigma + j\omega \tag{4.6}$$

Where:

σ and ω are respectively the real part value and the imaginary part value of the complex variable.

Substituting (4.6) in (4.5), we obtain:

$$z = e^{(\sigma + j\omega)T_s} = e^{(\sigma T_s + j\omega T_s)} = e^{(\sigma T_s)}.e^{(j\omega T_s)} \tag{4.7}$$

Using Euler's formula, we can write (4.7) as:

$$z = e^{(\sigma T_s)}.[\cos(\omega T_s) + j\sin(\omega T_s)] = e^{T_s\sigma}\lfloor \omega T_s = |z|\lfloor \omega T_s \tag{4.8}$$

We can distinguish three different cases regarding the value of the parameter σ. These are:

- **Case of $\boldsymbol{\sigma} < 0$**, which corresponds to the left hand side of the s-plane, and this gives:

  $$|z| = e^{T_s\sigma} < 1$$

  That is the left hand side of s-plane corresponds to the inside the unity circle of the z-plane.

- **Case of $\boldsymbol{\sigma} = \mathbf{0}$**, which corresponds to the imaginary axis of the s-plane, and this gives:

  $$|z| = e^{T_s\sigma} = 1$$

  That is the imaginary axis of s-plane corresponds to the boundary of the unity circle of the z-plane.

- **Case of $\boldsymbol{\sigma} > 0$**, which corresponds the right hand side of s-plane, and this gives:

  $$|z| = e^{T_s\sigma} > 1$$

  That is the right hand side of s-plane corresponds to the outside of the unity circle of the z-plane.

Note:

It is important to notice that the unit circle is defined in the complex z-plane as the circle of radius equals one ($r = 1$) and centered at the point (0,0).

From this demonstration we can establish a mapping between s-plane and z-plane. This mapping is illustrated in **Fig.4.1**.

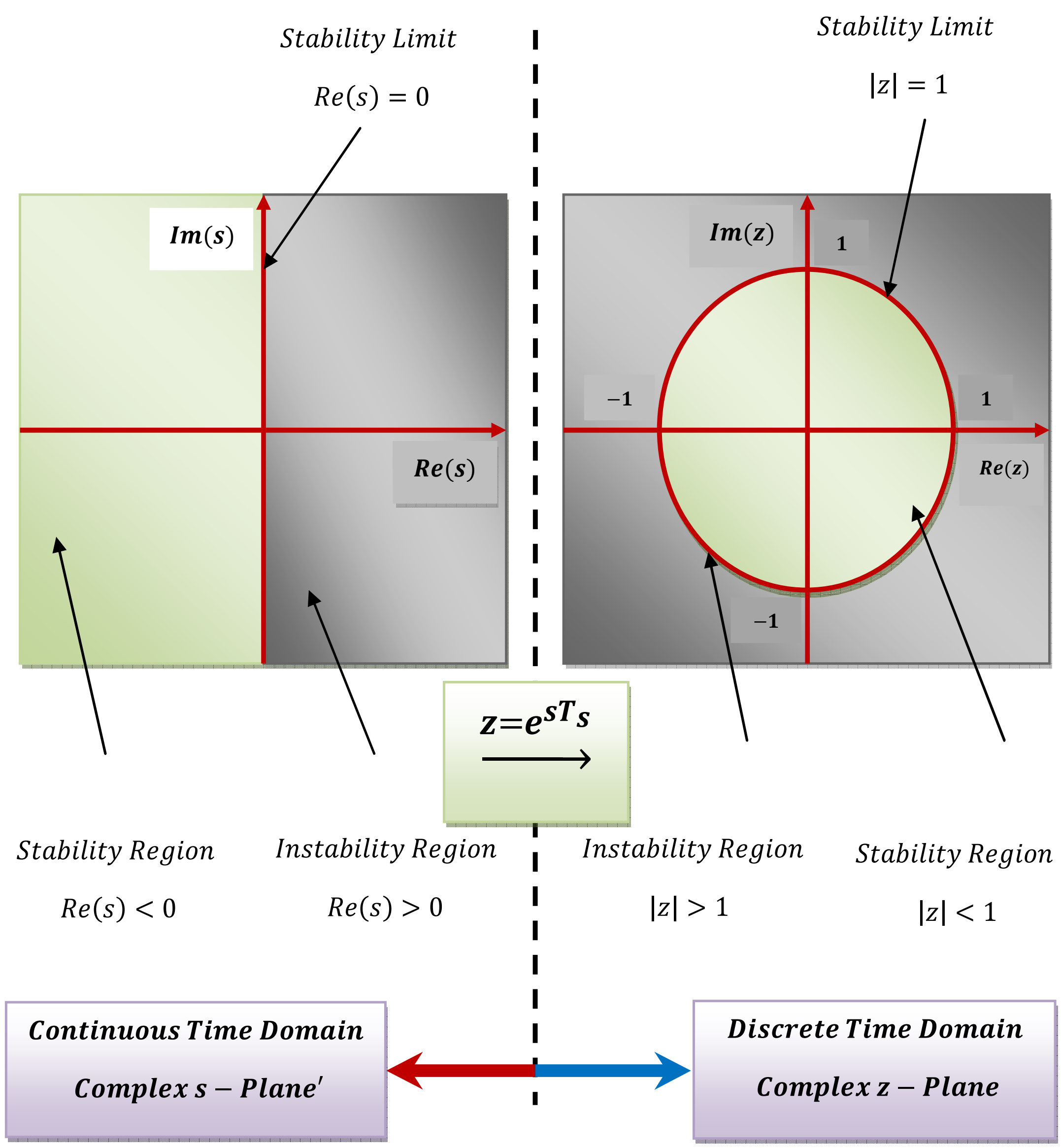

Fig.4.1 Mapping between s-plane and z-plane

1.3.2. Stability Theorem of discrete control system

From this mapping and correspondence between s-plane and z-plane, and by analogy with the analog feedback control system, the stability of linear time invariant discrete (sampled data) control system can be studied and analyzed according to the following results:

- The discrete feedback control system is said to be stable when all the poles of the corresponding discrete or z transfer function are located inside the unit circle of z-plane. In other words, for a discrete control system of order n, if

$p_i, i = 1,2,3, \dots \dots, n$ are its poles, then we say that this system is **stable** if and only if the following condition is satisfied:

$$\forall\, i, |p_i| < 1, \qquad i = 1,2,3, \dots \dots ., n$$

- The discrete feedback control system is said to be unstable when there exists at least one pole of its corresponding z transfer function is located outside the unit circle of the z-plane. For a discrete control system of order n, we express this as:

$$if\ \exists\ p_i, such\ that\ |p_i| > 1\ \Leftrightarrow the\ discrete\ system\ is\ unstable$$

1.3.3. Stability conditions based on discrete transfer function and pole location

The following stability conditions of discrete feedback control system which is represented by its z transfer function can be stated. The analysis and study of the different stability types can be easily done using these conditions and in accordance to the pole location.

1.3.3.1. Asymptotic and Marginal Stability

In order to determine whether a given discrete closed loop control system, which is represented by its discrete (also called) impulse transfer function is asymptotically or marginally stable, we accept without proof the following theorems.

Theorem 4.1: **Asymptotic and Marginal Stability**

In the absence of pole-zero cancellation, a Linear Time Invariant discrete (sampled data) control system is ***asymptotically stable*** if ***all*** of its z transfer function ***poles*** are located inside the unit circle of z-plane. If it exists ***at least one pole*** which is located on the boundary of the unit circle of z-plane, the so-called system is said to be ***marginally stable***. In the following example, we illustrate how to apply this theorem to test the asymptotic and marginal (critical) stability of discrete control systems.

1.3.3.2. Bounded Input Bounded Output (BIBO) Stability

Regarding the discrete control system which is represented by its z transfer function and based on the principle of pole location in the z-plane, we can also test whether it is bounded input bounded output stable. To do so, the following theorem states the necessary and sufficient condition for BIBO stability.

Theorem 4.2: Stability in the sense of BIBO

If there is a pole-zero cancellation, a discrete LTI closed loop control system is ***Bounded input-Bounded output*** stable if and only if ***all the poles*** of its corresponding z transfer function are located inside the unit circle.

1.3.3.3. Some Results and Discussion

The following notes can be pointed out in respect of the stability analysis of discrete control system.

- In the case of there is no pole-zero cancellation in the z transfer function representing the LTI discrete control system, ***asymptotic stability*** is identical to ***Bounded Input-Bounded Output stability***.
- These two theorems applied to determine the stability of discrete closed loop control system is also valid for the same purpose of discrete open loop control system.

Based on the above remarks and by combining the two theorems (4.1) and (4.2), we can conclude that even if there is a pole-zero cancellation, the asymptotic or marginal stability of the system can be concluded provided that the cancelled pole(s) is (are) stable; in other words, it (they) is (are located) inside or on the boundary of the unit circle of z-plane. This is true because stability determination is in fact the look for unstable poles. With the system declared stable, it means none is found. However, stable but hidden poles don't lead to wrong conclusion about the stability. On the other hand, hidden and unstable poles do lead to draw wrong conclusion. We can notice this important result in the following example.

Example 4.1

Consider the linear time invariant control systems represented by the discrete transfer functions as below:

1) $G_1(z) = \frac{4(z-2)}{(z-2)(z-0.1)}$

2) $G_2(z) = \frac{4(z-0.2)}{(z-0.2)(z-0.1)}$

3) $G_3(z) = \frac{5(z-0.3)}{(z-0.2)(z-0.1)}$

4) $G_4(z) = \frac{8(z-0.2)}{(z-1)(z-0.1)}$

We would like to determine whether these discrete transfer functions are asymptotically, marginally or Bounded Input-Bounded Output stable.

Answer 4.1

By referring to the two aforementioned theorems, the transfer functions $G_1(z)$ and $G_2(z)$ both of them present a pole-zero cancellation and their remaining poles are located inside the unit circle ($p_1 = 0.1, p_1 = 0.1$ *and* $|p_1 = 0.1| < 1$), hence these transfer functions are both BIBO stable.

Concerning the discrete control systems represented respectively by the transfer functions $G_3(z)$ and $G_4(z)$, it is obvious that there is no pole-zero cancellation. Since $G_3(z)$ have two poles $p_1 = 0.2, p_2 = 0.1$ *and* $|p_1 = 0.2| < 1, |p_2 = 0.1| < 1$; that is both of them are located inside the unit circle and $G_3(z)$ is ***asymptotically stable*** (theorem 4.1).

For the transfer function $G_4(z)$, it has two poles $p_1 = 1, p_2 = 0.1$ *and* $|p_1 = 1| = 1, |p_2 = 0.1| < 1$; that is one pole is located on the boundary of the unit circle which indicates that it is ***marginally*** (***critically***) **stable** (theorem 4.1).

As we can notice, the transfer function $G_1(z)$ although it contains one unstable hidden pole ($p_1 = 2$), it is however stable in sense of BIBO.

Consequently, BIBO stability is not always enough to judge and conclude about the absolute stability of discrete control system.

2. Stability Criteria applied to Discrete control systems

In this section, we will present and discuss some criteria used to judge and determine whether a discrete (digital) control system is stable or not by viewing and analyzing its open loop or closed loop z transfer function.

Several criteria (also called tests) are available and can be used to determine and test the stability of a discrete control system. Mostly used are:

- Jury stability criterion
- Routh Hurwitz stability criterion
- Nyquist stability criterion

2.1. Jury Criterion of Stability

Jury criterion of stability is an algebraic method which is used to evaluate and test the stability of a discrete LTI control system whose discrete transfer function is known. Determining the stability of the system using this criterion is based solely on the known coefficients of the characteristic polynomial of either the discrete open loop or closed loop transfer function.

If we consider that the discrete transfer function of the control system is expressed as a ration of the numerator and denominator polynomials as:

$$G(z) = \frac{Y(z)}{R(z)} = \frac{N(z)}{D(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots . + b_{m-1} z^1 + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots . + a_{n-1} z^1 + a_n}, \quad n \geq m, \qquad (4.9)$$

With:

$Y(z), R(z)$: are respectively the z transform of the input and output of the control system.

$a_0, a_1, \ldots., a_{n-1}, a_n$; $\quad b_0, b_1, \ldots., b_{m-1}, b_m$: are known real coefficients, where $a_0 > 0$ is a necessary condition for the use of Jury criterion.

In this transfer function, the characteristic polynomial is determined to be:

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots . + a_{n-1} z^1 + a_n \qquad (4.10)$$

2.1.1. Jury Table

The first step in using Jury criterion to evaluate and determine the stability of the control system is by constructing the so-called Jury table. This is done as mentioned below:

Table 4.1 Jury Table

line	z^0	z^1	z^2	z^3	… … … ..	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	… … … ..	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	… … … ..	a_{n-2}	a_{n-1}	a_n
3	B_{n-1}	B_{n-2}	B_{n-3}	B_{n-4}	… … … ..	B_1	B_0	
4	B_0	B_1	B_2	B_3	… … … ..	B_{n-2}	B_{n-1}	
5	C_{n-2}	C_{n-3}	C_{n-4}	C_{n-5}	… … … ..	C_0		
6	C_0	C_1	C_2	C_3	… … … ..	C_{n-2}		
. . .	. . .	. . .	. . .	. . .	. . .			
$\mathbf{2n-5}$	P_3	P_2	P_1	P_0				
$\mathbf{2n-4}$	P_0	P_1	P_2	P_3				
$\mathbf{2n-3}$	Q_2	Q_1	Q_0					

Where:

B_i, C_i, P_i, Q_i: are inserted real coefficients and are determined as:

$$B_i = \begin{vmatrix} a_n & a_{n-1-i} \\ a_0 & a_{i+1} \end{vmatrix}, \qquad i = 0,1,2,3, \dots \dots, n-1$$

$$C_i = \begin{vmatrix} B_{n-1} & B_{n-2-i} \\ B_0 & B_{i+1} \end{vmatrix}, \qquad i = 0,1,2,3, \dots \dots, n-2$$

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$$Q_i = \begin{vmatrix} P_3 & P_{2-i} \\ P_0 & P_{i+1} \end{vmatrix}, \qquad i = 0,1,2$$

2.1.2. Jury criterion statement

The use of Jury criterion as a tool to determine and direct test of the stability of the discrete control system is based on the satisfaction of the following corresponding conditions.

Consequently, a discrete control system which is characterized by its characteristic polynomial:

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots . + a_{n-1} z^1 + a_n \quad , \quad a_0 > 0$$

Is said to be stable if and only if all of the following conditions are simultaneously satisfied:

(1) $|a_n| < a_0$

(2) $D(1) > 0$

(3) $(-1)^n D(-1) > 0$

(4) $|B_{n-1}| > |B_0|$

(5) $|C_{n-2}| > |C_0|$

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$|Q_2| > |Q_0|$

2.1.3. Application procedure

Jury criterion of stability is a powerful tool for studying the stability of control systems. It allows us, however, to only determine whether the discrete open or closed loop control system is stable or not stable and no more information and details about, for instance, the pole locations. In order to correctly use and apply this tool, a basic procedure is indeed required.

First of all, we need to determine number of lines to be used in the Jury table. This depends on the order of the discrete control system under study; where we use the general formula given as:

$$2n - 3 \tag{4.11}$$

With $'n'$ represents the order of the control system.

After that, we construct the table as it is shown in Table 4.1, where the different unknown coefficients are calculated as it is indicated.

The last step is to check and verify the Jury conditions for stability; if all these conditions are simultaneously satisfied, the control system of interest is stable. But if at least one condition is not satisfied, the system will be judged unstable.

To illustrate the used of this procedure, we consider the following simple example.

Example 4.2

We assume that the discrete control system has the characteristic polynomial given by:

$$D(z) = z^2 + 9z + 8$$

Use Jury stability criterion and determine the stability of the system.

Answer 4.2

Firstly, we determine the number of lines of the Jury table.

We have:

$$n = 2, \qquad a_0 = 1, \qquad a_1 = 9, \qquad a_n = a_2 = 8$$

Which gives:

$$The\ number\ of\ lines = 2n - 3 = 1$$

Consequently, the Jury table corresponding to our discrete control system is as shown below:

$ligne$	z^0	z^1	z^2
1	a_2	a_1	a_0

After we have constructed the Jury table, we move to check and verify the stability conditions, we have:

(1) $|a_2 = 8| < 1$

(2) $D(1) = 1 + 9 + 8 = 18 > 0$

(3) $(-1)^2 D(-1) = 1 - 9 + 8 = 0 > 0$

We notice that conditions (1) and (3) are not satisfied, which lead us to conclude the instability of the given discrete control system.

Notice that:

In the above explanation of using Jury stability criterion, we have assumed that:

$$a_0 > 0$$

In case when $a_0 < 0$, the idea is :

- To obtain a new characteristic polynomial: $P(z) = -D(z)$
- Then, we apply Jury stability criterion with the polynomial $P(z)$.

2.2. Routh Hurwitz Stability Criterion

Routh Hurwitz method is another criterion widely used to determine and test the stability feedback control systems. However, this method is most familiar to be directly applied for the stability study and determination of continuous time control systems. In this vein, to be able of using this method for discrete control system, we need first to explain its use for exploring the stability of continuous time system.

Consider the continuous time feedback control system defined and described by transfer function in the s plane as:

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots . + b_1 s^1 + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots . + a_1 s^1 + a_0}, \quad n \geq m, a_n > 0 \qquad (4.12)$$

With:

$R(s), Y(s)$: are respectively are the Laplace transform of the input and the output of the control system.

We define the characteristic polynomial of the system's transfer function to be:

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots . + a_1 s^1 + a_0$$

The Routh Hurwitz criterion of stability uses the parameters of the characteristic polynomial to firstly construct the so-called Routh table as shown in **Table 4.2**:

Table 4.2 Routh Hurwitz stability table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	… … … ..
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	… … … ..
s^{n-2}	b_1	b_2	b_3	… … … ..	
s^{n-3}	c_1	c_2	c_3	… … … ..	
s^{n-4}	. . .	. . .	. . .		
s^1	j_1				
s^0	k_1				

The first two rows of the Routh table are formed by just listing the coefficients of the characteristic polynomial as it is shown.

For the new coefficients b_i and c_i which are inserted and entered in the subsequent rows of the table, they are calculated as follows:

$$b_1 = \frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}}{a_{n-1}}$$

$$b_2 = \frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix}}{a_{n-1}}$$

$$b_3 = \frac{\begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix}}{a_{n-1}}$$

The process of calculating the coefficients b_i should continue until the remaining values of $b's$ are all zero. When it is the case, we proceed to calculate the other inserted coefficient, that is:

$$c_1 = \frac{\begin{vmatrix} b_1 & b_2 \\ a_{n-1} & a_{n-3} \end{vmatrix}}{b_1}$$

$$c_2 = \frac{\begin{vmatrix} b_1 & b_3 \\ a_{n-1} & a_{n-5} \end{vmatrix}}{b_1}$$

$$c_3 = \frac{\begin{vmatrix} b_1 & b_4 \\ a_{n-1} & a_{n-7} \end{vmatrix}}{b_1}$$

Similarly, the process of calculating the coefficients c_i should continue until the remaining values of $c's$ are all zero.

The construction of the Routh table is continued in a similar manner until we finish with all the inserted and introduced coefficients where the Routh array will always be terminated with the two rows: s^1, s^0 which contain only respectively the two elements: j_1, k_1.

2.2.1 Statement of Routh Hurwitz Stability Criterion

Given a continuous time control system which is defined and represented by its transfer function $G(s)$, therefore, Routh criterion of stability states that ***all the roots of the characteristic polynomial of the transfer function (poles) have negative real parts if and only if the coefficients (elements) of the first column of the Routh table have the same sign. On the other hand, the number of roots of the characteristic polynomial (poles of $G(s)$) with positive real parts is equal to the number of sign changes occurred on the coefficients (elements) of the first column of Routh table***.

Consequently, the control system is stable if and only if all the coefficients (elements) of the first column of Routh table are of the same sign.

From the above statement, it is obvious that Routh-Hurwitz criterion allows us to just determine the stability of the control system by indicating whether all the poles of transfer function are stable; that is they are all located on the left hand side of the s-plane or some poles instead have positive real part (located on the right hand side of the s-plane).

This stability condition however cannot directly be applied to investigate the stability of discrete-time control systems. Fortunately and thanks of using the so-called the bilinear transformation, it is possible to explore the stability of discrete control system which is represented by its z transfer function, by making transformation from s-plane (also named as w-plane) to z-plane and vice-versa, hence we can apply Routh Hurwitz stability conditions. This bilinear transformation is defined by the following relationship between the continuous complex variable w and the discrete complex variable z:

$$z = \frac{1+w}{1-w} \Leftrightarrow w = \frac{z-1}{z+1} \qquad (4.13)$$

By performing this transformation, the inside of unit circle of z-plane is transformed to the left hand side (LHS) of the w-plane and the outside of the unit circle in z-plane is transformed to the right hand side (RHS) of w-plane. This transformation is illustrated in the following drawing of **Fig.4.2**.

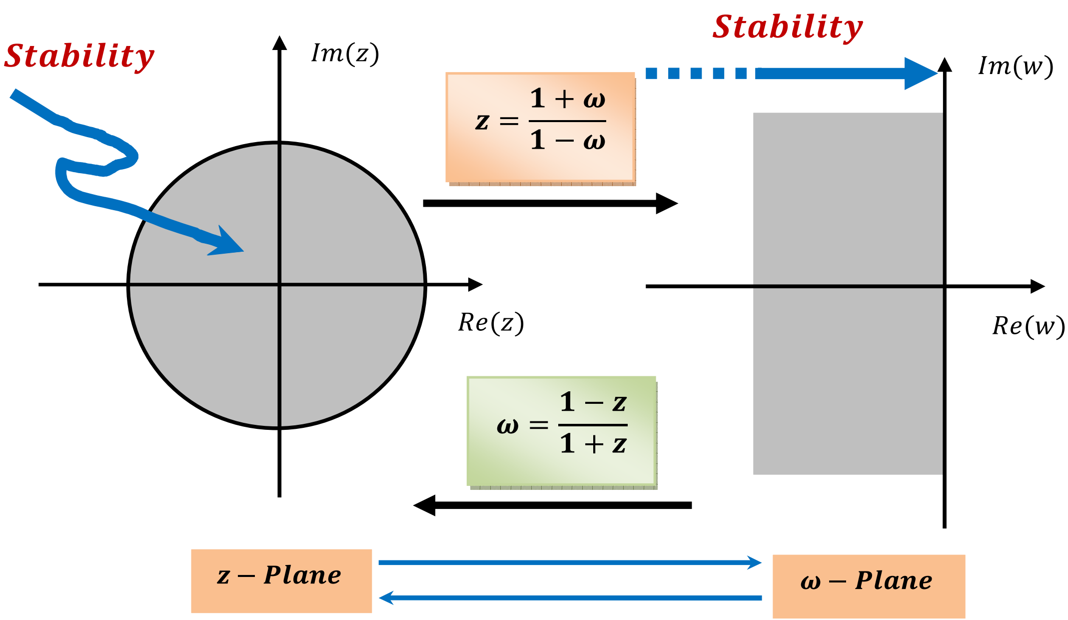

Fig. 4.2 Transforming z-plane into w-plane and vice-versa

Due to this transformation between the two planes, Routh Hurwitz criterion of stability is applied on the discrete control system as it is transformed from z-plane to w-plane. Therefore, the stability conclusion found in w-plane will be valid to judge the stability of the discrete control system in z-plane.

For the sake of illustration, we consider the following example.

2.2.2 Illustrative Example

Using Routh Hurwitz stability criterion, determine whether or not the system described by the following z transfer function is stable.

$$G(z) = \frac{2z+1}{z^3+2z^2+4z+7}$$

Solution:

In order to be able of applying Routh Hurwitz stability criterion on the above discrete control system represented by the z transfer function $G(z)$ for its stability investigation, we first use the bilinear transformation and transforming the study from z-plane to nw-plane.

$$z = \frac{w+1}{w-1}$$

By substituting in the transfer function, we obtain:

$$G(w) = \frac{2\left(\frac{w+1}{w-1}\right)+1}{\left(\frac{w+1}{w-1}\right)^3+2\left(\frac{w+1}{w-1}\right)^2+4\left(\frac{w+1}{w-1}\right)+7}$$

$$G(w) = \frac{\frac{2w+2+w-1}{w-1}}{\frac{(w+1)^3+2(w+1)^2(w-1)+4(w+1)(w-1)^2+7(w-1)^3}{(w-1)^3}}$$

$$G(w) = \frac{\frac{3w+1}{w-1}}{\frac{(w+1)[(w+1)^2+2(w+1)(w-1)+4(w-1)^2]+7(w-1)^3}{(w-1)^3}}$$

$$G(w) = \frac{(3w+1)(w-1)^2}{(w+1)[w^2+2w+1+2w^2-2+4w^2-8w+4]+7(w-1)^3}$$

$$G(w) = \frac{(3w+1)(w-1)^2}{(w+1)[7w^2-6w+3]+7(w-1)(w^2-2w+1)}$$

$$G(w) = \frac{(3w+1)(w-1)^2}{[7w^3-6w^2+3w+7w^2-6w+3]+[7w^3-21w^2+21w-7]}$$

$$G(w) = \frac{(3w+1)(w-1)^2}{14w^3-20w^2+18w-4}$$

Now, we construct the Routh table of the new transfer function $G(w)$ which is characterized by the polynomial:

$$D(w) = 14w^3-20w^2+18w-4$$

This gives:

w^3	14	18	0
w^2	-20	-4	0
w^1	b_1	b_2	b_3
w^0	c_1	c_2	c_3

$$b_1 = \frac{\begin{vmatrix} -20 & -4 \\ 14 & 18 \end{vmatrix}}{-20} = \frac{-360+56}{-20} = \frac{-304}{-20} = \frac{76}{5}$$

$$b_2 = \frac{\begin{vmatrix} -20 & 0 \\ 14 & 0 \end{vmatrix}}{-20} = 0$$

$$c_1 = \frac{\begin{vmatrix} \frac{76}{5} & 0 \\ 14 & 18 \end{vmatrix}}{\frac{76}{5}} = \frac{\frac{76}{5} . 18}{\frac{76}{5}} = 18$$

$$c_2 = \frac{\begin{vmatrix} \frac{76}{5} & 0 \\ 14 & 0 \end{vmatrix}}{\frac{76}{5}} = 0$$

It results the following final Routh table:

w^3	14	18	0
w^2	-20	-4	0
w^1	$\frac{76}{5}$	0	0
w^0	18	0	0

By observing only the first column of the table we notice that:

- Not all the elements (coefficients) of the first column of the array have the same sign. This leads us to conclude that the discrete control system is unstable.
- The number of sign changes among the coefficients of the first column equals two (02), which means that two (02) poles of the discrete transfer function are unstable.

2.2.3 Properties of Routh Hurwitz Stability criterion

From the theoretical and practical point of view of using Routh Hurwitz criterion to determine and investigate the stability of feedback control systems, we can point out the following properties:

1) The use of this stability criterion is completely independent of the order of the system's transfer function. In other words, it is applicable whatever the order of the system but it is particularly more useful for higher order systems.

2) This stability criterion can only tell us whether the control system is stable or not by observing the coefficients of first column of the table. Therefore, no information about the degree of stability (stability margins) can be drawn and concluded.
3) This criterion is not applicable to investigate the stability of feedback control systems which involve time delays.
4) As the last property of Routh Hurwitz stability criterion is that a necessary but not sufficient condition for the poles of the control system transfer function to be all with negative real parts is that all the coefficients of the characteristic polynomial $D(w)$ are of the same sign (all negative or all positive).

As a conclusion to this section and regarding the study and direct determination of the stability of feedback discrete (sampled data) control systems which are represented by a z transfer function, we have explored and discussed the two stability criteria of Jury and Routh Hurwitz as most familiar and widely used methods by the automatic control systems designers and engineers. Also these two methods (criteria) are simple and of straight forward use. Nevertheless, we can find other methods and criteria to test the stability of discrete control systems such as Nyquist method, Bode plot method and others. These methods however are design methods not just direct test of the stability of feedback control systems.

3. Accuracy Analysis of Linear sampled data control systems

The accuracy property is an important performance measure regarding the design and analysis of any feedback control system in general, and in particular for the discrete time control systems due the inherent characteristics caused after the sampling operation and the sampling period [6]. in a given feedback control system, the accuracy performance is measured and analyzed according to the value of the tracking error of the system's response, which is defined, in the time domain, as the difference between the reference signal (desired response), $r(k)$, and the actual measured output, $y(k)$. mathematically, we express this error as:

$$e(k) = r(k) - y(k) \tag{4.14}$$

Where:

$e(k)$: denotes the tracking error.

Regarding the accuracy of a feedback control system, two types of accuracy performance can be distinguished; namely:

- **Transient (dynamic) accuracy**: which characterize the accuracy behavior of the discrete time control system within the transient state of the system's response.
- **Steady state accuracy**: This describes the accuracy behavior of the discrete time control system at the steady state of the system's response.

We will be interesting of steady state accuracy due to its importance of giving a clear idea about the stability of the control system.

The relationship between the accuracy and the stability of discrete time feedback control system can be described by the following results:

- $\boldsymbol{The\ Control\ System\ is\ Accurate} \Longrightarrow \boldsymbol{it\ is\ Stable} \Leftrightarrow \forall\, \boldsymbol{p_i} / \, |\boldsymbol{p_i}| < 1$
- $\boldsymbol{The\ Control\ System\ is\ Unstable} \Leftrightarrow \exists\, \boldsymbol{p_i} / \, |\boldsymbol{p_i}| > 1 \Longrightarrow \boldsymbol{it\ is\ inaccurate}$

With:

$p_i: (i = 1,2,3, \ldots, n)$, are the 'n' poles of the z transfer function.

3.1. Steady state Accuracy

The steady state accuracy corresponds to steady state error of the feedback control system. In order to calculate the steady state error, we need to work out the following typical block diagram of a general discrete time unity feedback control system shown in **Fig.4.3**.

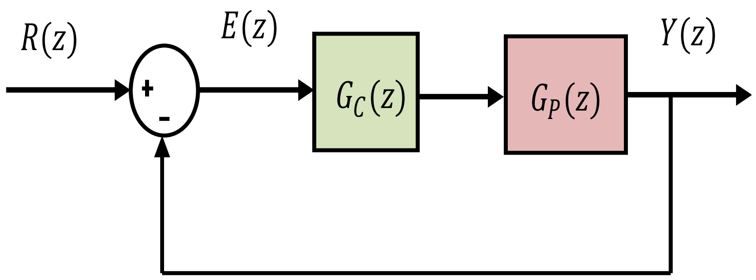

Fig.4.3 Typical discrete unity feedback control system

With:

$R(z), Y(z)$ and $E(z)$ are respectively the reference signal, the output signal and the tracking error signal, all expressed in frequency domain.

$G_C(z)$ and $G_P(z)$ are respectively the controller and the controlled process transfer functions.

We now proceed to calculate the tracking error as follows:

We have:

$$E(z) = R(z) - Y(z) = R(z) - G_C(z)G_P(z)E(z) \Longrightarrow (1 + G_C(z)G_P(z))E(z) = R(z)$$

That is:

$$E(z) = \frac{1}{1 + G_C(z)G_P(z)} R(z) \tag{4.15}$$

The steady state error corresponds to the error $e(k)$ as $k \to \infty$. Using final value theorem discussed in chapter 2, we can write:

$$e_{ss} = e(\infty) \lim_{k\to\infty} e(k) = \lim_{z\to 1}(1 - z^{-1})E(z) \tag{4.16}$$

Substituting (4.15) in (4.16), we get:

$$e_{ss} = \lim_{z\to 1}(1 - z^{-1}) \frac{1}{1 + G_C(z)G_P(z)} R(z) \tag{4.17}$$

If we define:

$$\frac{1}{1 + G_C(z)G_P(z)} = \frac{(z-1)^\alpha}{(z-1)^\alpha + K} \tag{4.18}$$

Where:

α and K represent respectively the order of astatism (also known as the class and the type of the feedback control system) and open loop static gain of the system.

Using (4.18), (4.17) can be rewritten as:

$$e_{ss} = \lim_{z \to 1} \left(\frac{z-1}{z}\right) \frac{(z-1)^{\alpha}}{(z-1)^{\alpha} + K} R(z) \tag{4.19}$$

From (4.19), it is obvious that the steady state error and hence the accuracy of the control system depends on the following parameters:

- The reference (Setpoint) signal, $R(z)$.
- The order of Astatism (or the type) of the control system, α.
- The open loop gain, K.

All these three parameters can be used to analyze and enhance the accuracy performance of the digital control system as it will be mentioned in the subsequent subsections.

3.2. Steady state accuracy due to input reference signal

The accuracy of the discrete feedback control system depends on the type of the input reference signal. As a result, it can be studied and analyzed according to three standard and basic reference signals as follows:

Table 4.3 different types of steady state errors corresponding to the types of input reference signals

Type of input reference signal		**Generated Steady State Error**
Step	$R(z) = \dfrac{z}{z-1}$	Position steady state error $(\boldsymbol{e_p}(\infty))$
Ramp	$R(z) = \dfrac{zT_s}{(z-1)^2}$	Velocity steady state error $(\boldsymbol{e_v}(\infty))$
Parabola	$R(z) = \dfrac{T^2 z(z+1)}{(z-1)^3}$	Acceleration steady state error $(\boldsymbol{e_a}(\infty))$

With: T_s is the sampling time period (second).

The following table (**Table 4.4**) summarizes the values of the steady state errors and consequently the accuracy performance study of the discrete time control system regarding the three parameters; namely the type of the input reference signal as well as the type and the open loop static gain of the control system.

Table 4.4 Values of steady state error depending on control system's order of Astatism, input reference signal and open loop gain

		Order of Astatism of the Control System			
		$\boldsymbol{\alpha = 0}$	$\boldsymbol{\alpha = 1}$	$\boldsymbol{\alpha = 2}$	$\boldsymbol{\alpha} > 0$
Steady State Errors $(\boldsymbol{e_{ss}})$	$\boldsymbol{e_p}(\infty)$	$\frac{1}{1+K}$	0	0	0
	$\boldsymbol{e_v}(\infty)$	∞	$\frac{T_s}{K}$	0	0
	$\boldsymbol{e_a}(\infty)$	∞	∞	$\frac{{T_s}^2}{K}$	0