Chapter 3: Modeling and Representation of Sampled Data Linear Control Systems

Representation and modeling is a mandatory task throughout the process of designing any feedback control system and analyzing its performance. It allows defining and establishing the relationship between mainly the input and output signal as well as other signals affecting the operating performance of the control system. In this chapter we will be interesting of explaining the different methods and approaches used to mathematically represent and model the behavior of the sampled data (discrete time) control system.

1. Basic Notions and Definitions

Before we tackle the subject of this chapter stated earlier, it is important to present some preliminary definitions which need to be known.

1.1. Automatic System

Automatic system is the field and domain that concerns the design methods and approaches which ensure the control of a physical system (process) behavior without the intervention of human being. Consequently, automatics deals with aspects like: modeling, identification and control system dynamics.

1.2. System Modeling

Modeling process can be defined as being the establishment and representation of the relationship between the input and the output variables as well as other affecting signals for a given physical system. The objective is to emphasize how a target output variable(s) can be controlled regarding the influence of input design or disturbance variables; hence mimicking the real behavior of the physical system.

1.3. System Identification

Identification of systems is mutually linked to its modeling, indeed it is the process of determining the designation of different variables describing the system's model as well as the values of the associated parameters. Modeling and identification are two necessary phases before any work of control that can be done for an automatic system.

1.4. System Control

Controlling a system is defined to be the use and application of a control law or mechanism to affect the variation of its output response with respect to the applied desired behavior and taking into account the influence of surrounding environment. The objective of system control is to ensure the desired performance of stability, robustness and other properties.

Regarding system control objectives, the general case is to make the system's output response follow and track the variable profile imposed at its input reference whatever the surrounding influencing conditions (such as: noise, disturbance,…etc.). Particular case is when the aim is to keep and maintain the output response constant and fixe at that value preset at the input reference in all operating conditions; this control system is known, in the field of automatics, as regulatory control (automatic) system.

Either general control system or regulatory control system, the following two types can be distinguished and implemented; we speak about open loop and closed loop control systems.

1.4.1. Open loop control system

An open loop control system is defined as the system when the control law does not depend on the value of the output but only on the value imposed at the input reference. In other word, an open loop control system there is no feedback from the output to the input.

For sampled data control system, a typical block diagram of open loop system is shown in **Fig.3.1**.

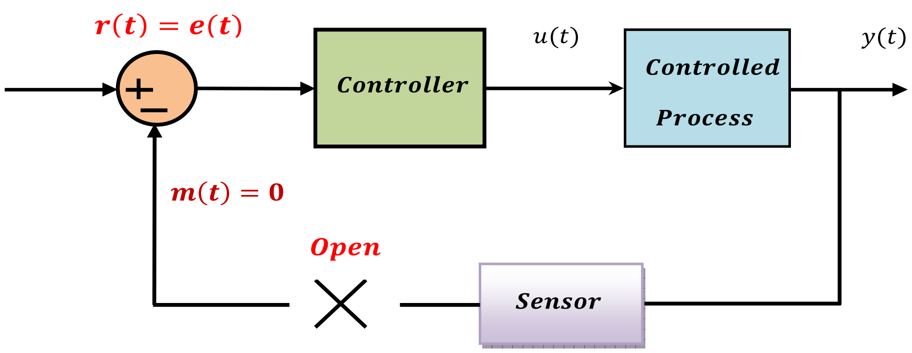


Fig.3.1 typical block diagram of open loop control system

1.4.2. Closed (feedback) loop control system

Unlike open loop control system, closed loop (also called feedback) control system is characterized by the fact that the control law depends on the actual value of the output response. This is implemented by feeding back the system's output to be compared with the imposed reference signal, which based on the comparison value the control and actuating signal is issued.

For sampled data control system, a typical block diagram of closed loop system is shown in **Fig.3.2**.

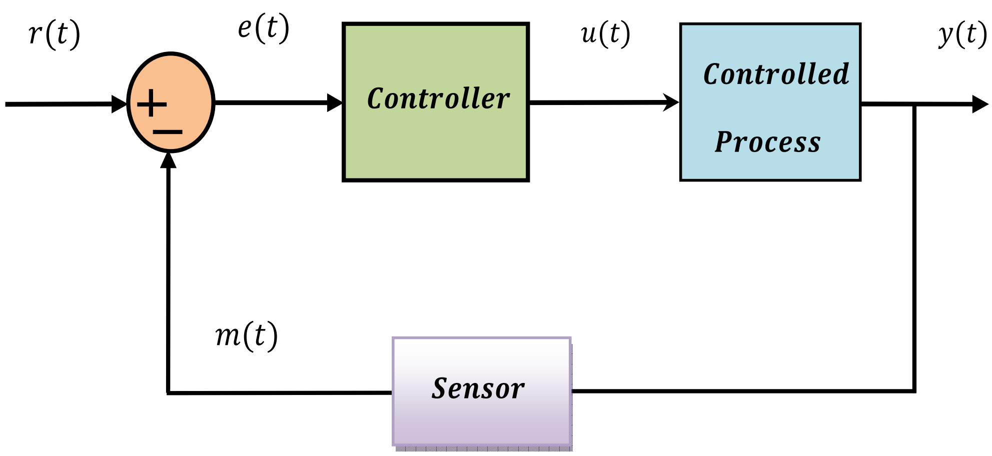


Fig.3.2 typical block diagram of closed loop control system

1.5. Continuous control system

A control system is said to be continuous (or analog) if both its input and output signals are continuous time or analog.

1.6. Sampled date control system

A control system is said to be sampled data, also called discrete or digital, if when it receives sampled (discrete or digital) signal it responds with a sampled or discrete time signal.

It is important to state that, in automatic control system domain, the terms sampled data, discrete and digital are equivalently and interchangeably used and employed.

1.7. Linear control system

A linear control system is defined as a system where its input-output representation or modeling can be described by a linear differential equation (for continuous time case) and by a linear difference equation (for discrete time case). This means that, for linear system, the system parameters do not vary as a function of signal level [3].

1.8. Time invariant control system

Generally we define a time invariant system as being the system for which when a time shifted input signal is applied, the corresponding produced and generated output response signal is characterized by the same amount of time shift.

1.9. Linear time invariant (LTI) control system

A control system is said to be linear time invariant (LTI) when it is both linear and time invariant. From the mathematical point of view, an LTI system can be represented and modeled using linear differential equation of constant coefficients (for the case of continuous LTI system) or using difference equations (for the case of discrete LTI system).

1.10. SISO and MIMO control systems

Single input single output (SISO) control systems are characterized by only one signal as input and one controlled signal as output, however, multi-input multi-output (MIMO) control systems are those systems which are characterized by several variable signals as inputs and several controlled variable signals as outputs. It is important to point out that the same control methodology can be applied for both SISO and MIMO systems, the difference resides in the modeling and identification phases as well as the fact that MIMO systems encompass many output variables need to be controlled in response to the applied inputs.

2. General block diagram of sampled data control system

A sampled data (discrete) control system is implemented around a digital controller instead of the old existing analog controller; that is we may need only to change the analog controller and keep all other parts of the control system such as the controlled process and the used sensors. However, for the digital controller, as a digital system, to work and operate properly, additional signal conversion devices need to be inserted at its input and output for signal conditioning and adaptation purposes.

Among these devices to be used with the digital controller are respectively the Analog-to-Digital Converter (ADC) at the controller's input and the Digital-to-Analog Converter (DAC) placed at the controller's output.

With these two conversion devices inserted additionally with the digital controller, the general structure of sampled data (discrete) control system becomes as it is shown in **Fig.3.3**.

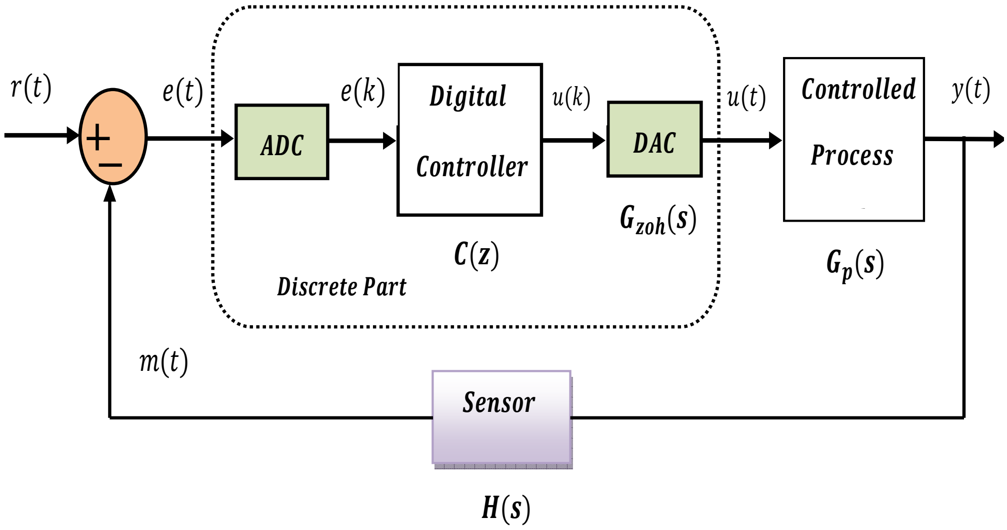


Fig.3.3 General block diagram of sampled data (discrete) control system

2.1. Component description of typical sampled data control system

From the general and typical block diagram representing the sampled data (discrete) control system, the following components and parts are generally found.

- **Analog Process or Plant** ($G_p(s)$): this represents the process to be controlled; that is it involves the output variable(s) being controlled or regulated. The controlled process is often preceded an actuating element or actuator.
- **Digital Controller** ($C(z)$)**:** this is the main component in a feedback control system which is compulsorily being discrete or digital. The digital controller is usually represented by a discrete transfer function denoted as: $C(z)$.

- **Sensor** ($H(s)$): for feedback control system we always need to measure the actual value of the output variable subjected to control. This is achieved using the sensor, which is generally represented by the s-transfer function $H(s)$.
- **Analog-to-Digital Converter (ADC):** in sampled data control system, the DAC is necessary to convert the analog signal into digital one and hence make it pertinent and adequate as input to the digital controller of the system.
- **Digital-to-Analog Converter (DAC):** the output of the digital controller is discrete (digital) which is applied as input to control the output variable of the analog process. As an analog system, the controlled process requires, for its proper operation and functioning, to be actuated by an analog input. Due to that fact, a DAC is inserted after the digital controller to convert its output digital signal into analog signal suitable as input to the process.

Besides these components which are inherent when implementing any sampled data (or digital) control system, the following signal nomenclature is also necessarily to be known. We mean:

- **Setpoint signal ($\boldsymbol{r(t)}$):** it represents the reference real valued quantity to be reached or followed by the controlled output variable of the control system. For general sampled data control system, this signal is assumed continuous time (analog).
- **Controlled output signal ($\boldsymbol{y(t)}$):** this is the output variable of the control system; more precisely, it is the output of the process targeted by the control or regulation. In major cases of implemented sampled data control system, the controlled process is of type analog; hence is its controlled output variable.
- **Measured output signal ($\boldsymbol{m(t)}$)**: this signal gives us a quantitative measure of the actual value of the controlled output variable $y(t)$ which is issued by used sensor. The measured output signal may be discrete time or continuous time depending on the implementing method of the control system;

however, in our particular case of system's block diagram shown in **Fig.3.3**, it is assumed analog-type signal.

- **Error signal** ($\boldsymbol{e(t)}$): this signal represents instantly the difference between the setpoint and the measured controlled output signal values. Mathematically, we define the continuous time type error signal as:

$$e(t) = r(t) - m(t) \tag{3.1}$$

For a particular case of unity feedback control system, we have $m(t) = y(t)$, expression (3.1) becomes:

$$e(t) = r(t) - y(t) \tag{3.2}$$

Obviously, the type of the error signal whether it is discrete time or continuous time depends on both the sensor's technology used (analog or digital) and the implemented sampled data control system.

- **Control (actuating) signal ($\boldsymbol{u(k)}$):** it is the most important quantity in any feedback control system (continuous or discrete), which handles the action taken by the controller and being applied on the controlled process to alter the output response. Of course this signal is originally discrete (digital) because it is outputted by the used digital controller; it is being instantly converted into continuous time quantity $u(t)$ using the inserted DAC to best suit the proper operation and functioning of the controlled process.

Regarding its importance in the control system's general block diagram and in order to include it in further analysis of the control system performance and characteristics, a modeled and representative component of the ADC converter is instead used. We will adopt the model represented and shown in **Fig.3.4**.

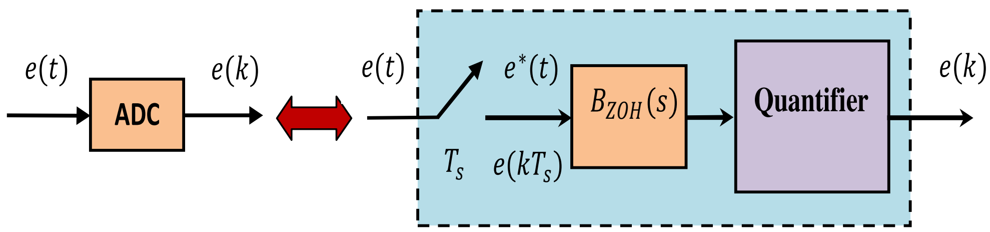


Fig.3.4 Typical model representation of ADC (sampler)

As the case of DAC, the ADC converter should also be given a model such that it can be included in the analysis and design of the sampled data control system. a typical model and representation of this component is shown in **Fig.3.5**.

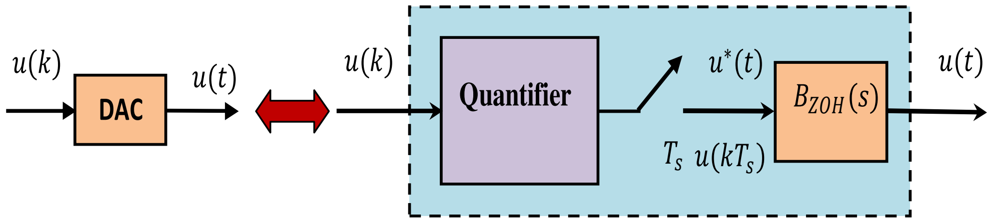


Fig.3.5 Typical model representation of DAC

Regarding the design and analysis of a typical sampled data (digital) control system, which is our fundamental focus and interest in this material, the quantifier in both ADC and DAC elements can be disregarded without affecting the targeted objectives. As such, the final models that can be given for these two components are represented in the following **Fig.3.6** and **Fig.3.7** respectively.

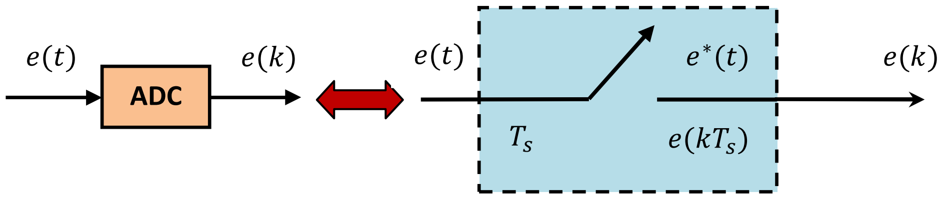


Fig.3.6 Final model representation of ADC (sampler)

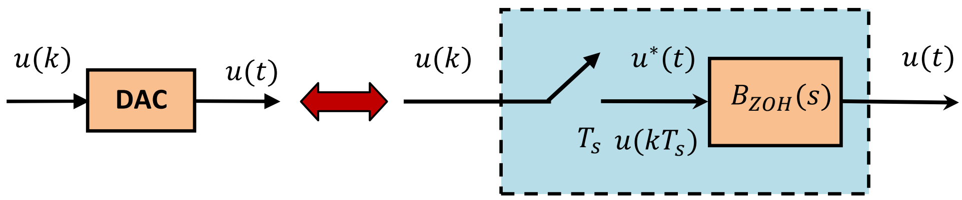


Fig.3.7 Final model representation of DAC

Consequently, the final general block diagram representing a sampled data control system shown earlier in Fig.3.3 is typically represented by the block diagram of **Fig.3.8**.

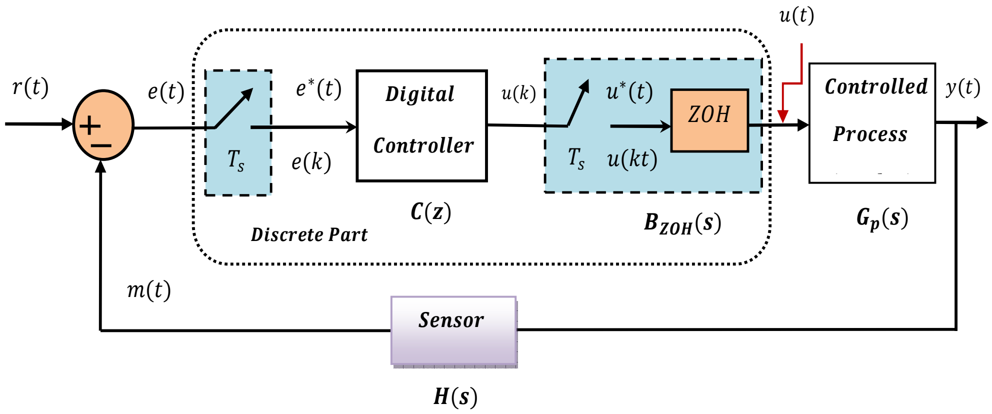


Fig.3.8 Typical general block diagram representation of sampled data control system

3. Sampled data linear time invariant (LTI) control system modeling and representation

3.1. Modeling of Linear time invariant system

For any sampled data linear time invariant (LTI) system which is characterized by a single input and single output (SISO), a general input-output representation can be given by the following diagram.

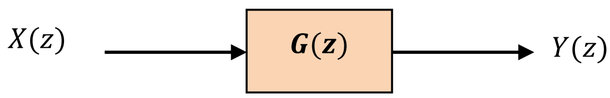


Fig. 3.9 General block diagram representation of LTI SISO and sampled data system

With:

$X(z), Y(z)$: represent, respectively the systems' input and output.

$G(z)$: denotes the transfer function describing the system's dynamic behaviour.

To model and represent the dynamic behavior of a sampled data (discrete time) system, we distinguish three methods:

- Discrete transfer function.

- Difference equations.
- Block diagram representation.

The following sections of this chapter will devoted to explain and describe how a discrete time system can be modeled and represented using these methods, where the relationship between the different representations is mentioned as well.

3.2.1 Modeling using discrete transfer function

As in the case of continuous time system, the discrete time system can also be represented by a discrete transfer function. Mathematically, it is defined as the ratio between the z transform of the input and the output of the system. the discrete transfer function is denoted by $G(z)$ and is expressed as:

$$G(z) = \frac{Y(z)}{X(z)} \tag{3.3}$$

With:

$X(z)$ and $Y(z)$ represent respectively the z transform of the input and the z transform of the output.

Analytically, the discrete transfer function of a sampled data system can be explicitly expressed under three forms:

- Polynomial ratio as a function of the variable 'z'.
- Polynomial ratio as a function of the variable 'z^{-1}'.
- Gain-pole-zero form.

(1) Polynomial ratio as a function of the variable 'z'

A general form and expression of the discrete transfer function using the ratio of two polynomials as a function of the complex variable 'z' is given as follows:

$$G(z) = \frac{N(z)}{D(z)} = \frac{(b_0 z^m + b_1 z^{m-1} + \cdots + b_m z^0)}{(a_0 z^n + a_1 z^{n-1} + \cdots + a_n z^0)} \tag{3.4}$$

With:

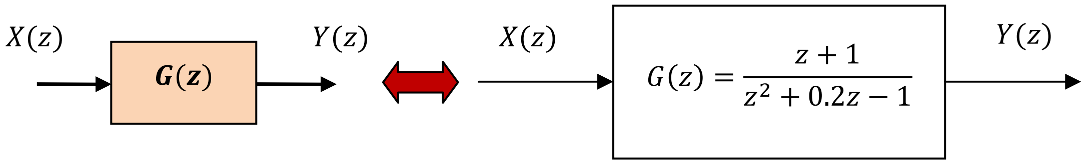
$N(z)$ and $D(z)$ are respectively called the numerator and denominator of the transfer function $G(z)$.

b_j and a_i $(i = 0,1,2,3,\ldots,n, j = 0,1,2,3,\ldots,m)$ are known coefficients corresponding respectively to the numerator $N(z)$ and denominator $D(z)$.

m and n are respectively the degrees of the numerator and denominator of the transfer function.

Example:

As an example of this representation form of a discrete transfer function, we consider the following:



In this example, we have: $N(z) = z + 1$, $D(z) = z^2 + 0.2z - 1$, where we notice that both polynomials are function of the complex variable 'z'.

We can use Matlab environment to implement this form of the discrete transfer function using the Matlab user defined function '***tf***'. this is done as follows:

```
num_G =[1 1];
den_G =[1 0.2 -1];
Ts = 1;
G = tf(num_G, den_G,Ts,'variable','z')
```

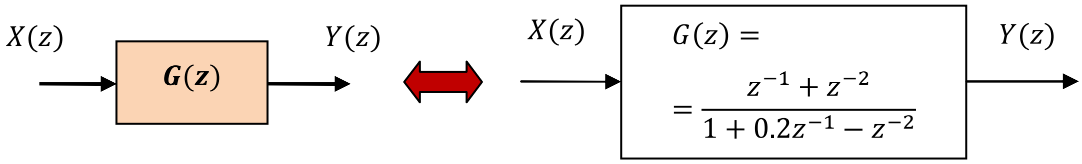
(2) Polynomial ratio as a function of the variable 'z^{-1}'

In some cases and for the sake of making easy the analytical manipulation and analysis of the discrete (sampled data) control system, we often require that the complex variable be 'z^{-1}' instead of 'z'. Therefore, by expressing the numerator $N(z)$ and the denominator $D(z)$ as a function of the complex variable 'z^{-1}', the discrete transfer function can also be given the following form;

$$G(z) = \frac{N(z)}{D(z)} = \frac{(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})}{(a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})} \tag{3.5}$$

Example;

As an example to the discrete transfer function expressed under this form we give:



For which, $N(z) = z^{-1} + z^{-2}$, $D(z) = 1 + 0.2z^{-1} - z^{-2}$, and both of the polynomials are function of 'z^{-1}'.

In this case also, it is possible to implement the discrete transfer function in Matlab environment using the following set of instruction with Matlab function '**tf**':

```
num_G=[0 1 1];
den_G=[1 0.2 -1];
Ts= 1;
G = tf(num_G, den_G,Ts,'variable','z^-1')
```

(3) Gain-pole-zero form of a discrete transfer function

Sometimes, it is helpful to factorize the numerator $N(z)$ and denominator $D(z)$ with respect to their roots. The result will be a new form of the discrete transfer function known as Gain-pole-zero form that is expressed as follows:

$$G(z) = \frac{N(z)}{D(z)} = \frac{b_0}{a_0}\frac{(z-z_1)(z-z_2)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_n)} = \frac{b_0}{a_0}\frac{\prod_{j=1}^{m}(z-z_j)}{\prod_{i=1}^{n}(z-p_i)} \tag{3.6}$$

$$G(z) = \frac{N(z)}{D(z)} = \frac{b_0}{a_0}\frac{(1-z_1z^{-1})\dots(1-z_mz^{-1})}{(1-p_1z^{-1})\dots(1-p_nz^{-1})} = \frac{b_0}{a_0}\frac{\prod_{j=1}^{m}(1-z_jz^{-1})}{\prod_{i=1}^{n}(1-p_iz^{-1})} \tag{3.7}$$

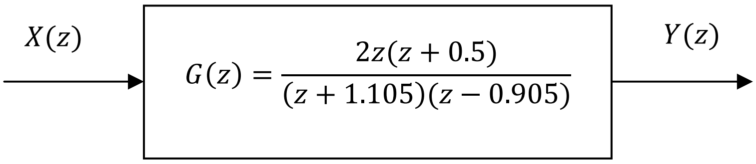
With:

$\frac{b_0}{a_0}$: represents the gain of the discrete system.

p_i and z_j (with $i = 1,2,3,\dots,n$, $j = 1,2,3,\dots,m$) represent respectively the roots of the polynomials $D(z)$ and $N(z)$, which are respectively known to be the poles and zeros of the discrete transfer function representing the discrete system.

Example:

As an example of the discrete transfer function written under the form gain-pole-zero, we give:



Obviously, in order to express a discrete transfer function under the form gain-pole-zero, from the polynomial ratio, we follow the steps in below:

- Determine and calculate the gain of the transfer function using its definition as:

$$K = \frac{b_0}{a_0}$$

- Determine the poles p_i and zeros z_j of the transfer function which are respectively the roots of the polynomials $D(z)$ and $N(z)$.
- Finally, the gain-pole-zero of the transfer function is either of the forms (3.6) or (3.7).

We can also generate the gain-pole-zero form of the discrete transfer function using MATLAB environment via the set of the following instructions.

```
Zeros_G= [0 -0.5]' ; Poles_G= [-1.105 0.905]';
K= 2; Ts= 1;
G = zpk(Zeros_G ,Poles_G ,K ,Ts)
```

In this Matlab script, the gain of the discrete transfer function is determined to be equals $K = \frac{b_0}{a_0} = 2$.

On the other hand, the poles and zeros of the transfer function are respectively given by the two row vectors as follows:

$$p_i = [-1.105 \quad 0.905]^t = \begin{bmatrix} -0.105 \\ 0.905 \end{bmatrix} \quad ; \qquad z_j = [0 \quad -0.5]^t = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$$

By using the MATLAB user-defined function '**zpk**', we generate the gain-pole-zero form of the transfer function.

3.1.1.1. System Characteristics from Discrete Transfer Function

Regardless the form under which the discrete transfer function can be expressed, the following properties and characteristics of the digital control system are being extracted and deduced.

(i) The roots of the numerator $N(z)$ are called the **zeros** of the transfer function (or of the system). They are denoted as: $z_j, j = 1,2,3, \dots, m$.

(ii) The roots of the denominator $D(z)$ are called the **poles** of the transfer function (or of the system). They are denoted as: $p_i, i = 1,2,3, \dots, n$.

(iii) The denominator $D(z)$ is called the **characteristic polynomial** of the discrete transfer function.

(iv) Correspondingly, the equation defined as $D(z) = 0$ is known as the **characteristic equation** of the discrete system.

(v) The **degree** of the polynomial $D(z)$ is equal to **the order** of transfer function (or the discrete system).

(vi) Lastly, for the discrete system (transfer function) to be **realizable**, it should satisfy the condition of $n \geq m$

3.2.2 Modeling using Difference Equation

The difference (also called recurrent) equation for linear time invariant (LTI) sampled data (discrete time) system is analogous to differential equation used to describe and model the LTI analog (continuous time) system.

In general form, the dynamics of an LTI discrete system can be described by the difference equation given as:

$$\sum_{j=0}^{m} b_j x(k-j) = \sum_{i=0}^{n} a_i y(k-i) \qquad (3.8)$$

With :

$x(k-j)$ and $y(k-i)$; $i = 0,1,,2, \dots, n$; $j = 0,1,2,3, \dots, m$ are respectively the $(m+1)$ input samples and $(n+1)$ output samples of the discrete system.

b_j, a_i: are assumed to be known real coefficients.

Therefore, using the difference equation, we can dynamically model the behaviour of the discrete system via the time occurrence of the input and output samples.

As a particular case, when $a_i = 1,\ i = 0$, the equation (3.8) is also written as:

$$y(k) = \sum_{j=0}^{m} b_j x(k-j) - \sum_{i=1}^{n} a_i y(k-i) \qquad (3.9)$$

The equation (3.9) is particularly simpler when implementing the discrete time (digital) system.

3.2.2.1 Relationship between transfer function and difference equation representation

The two representations of transfer function and difference equation used to model a discrete control system are in fact interrelated; that is we can go from one representation to another, also the reverse operation of obtaining one representation from the other is possible.

In the following, we will describe, in general manner, how this relationship is established.

We consider that we have a discrete system represented by its z transfer function given generally as:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{N(z)}{D(z)} = \frac{(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})}{(a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})} \qquad (3.10)$$

With:

$Y(z)$ and $U(z)$ are respectively the Z transforms of the input and output of the discrete system.

For the sake of simplifying things, the two polynomials of the transfer function $N(z)$ and $D(z)$ are expressed as a function of z^{-1}.

By performing cross multiplication of (3.10), we obtain:

$$\frac{Y(z)}{U(z)} = \frac{(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})}{(a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})}$$

$$\Longrightarrow\ (a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})\, Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})\, U(z)$$

$$\Longrightarrow\ \left(a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_n z^{-n} Y(z)\right) =$$

$$= \left(b_0 U(z) + b_1 z^{-1} U(z) + \cdots + b_m z^{-m} U(z)\right) \qquad (3.11)$$

We apply next the inverse Z transform on both sides of expression (3.11), we get:

$$Z^{-1}\{a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_n z^{-n} Y(z)\} =$$

$$= Z^{-1}\{b_0 U(z) + b_1 z^{-1} U(z) + \cdots + b_m z^{-m} U(z)\}$$

Using Z transform and inverse Z transform properties, which are identical, particularly the two properties of linearity and Z transform of time delay, it results:

$$a_0 Z^{-1}\{Y(z)\} + a_1 Z^{-1}\{z^{-1} Y(z)\} + \cdots + a_n Z^{-1}\{z^{-n} Y(z)\} =$$

$$= b_0 Z^{-1}\{U(z)\} + b_1 Z^{-1}\{z^{-1} U(z)\} + \cdots + b_m Z^{-1}\{z^{-m} U(z)\}$$

$$\Longrightarrow a_0 y(k) + a_1 y(k-1) + \cdots + a_n y(k-n)$$

$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$$

Which is the corresponding difference equation representing the discrete (sampled data) system as it is obtained from the transfer function representation. This difference equation can be rewritten in more general form as:

$$\sum_{j=0}^{m} b_j x(k-j) = \sum_{i=0}^{n} a_i y(k-i) \qquad (3.12)$$

Obviously, equation (3.12) is equivalent to equation (3.8).

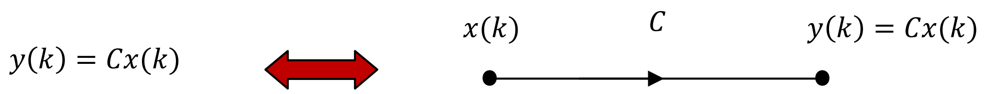
3.2.3 Modeling discrete system using block diagram

The representation and modeling of LTI discrete system using block diagram, three basic elements are indeed required to be introduced. These are:

- Scalar multiplication.
- Algebraic sum
- Discrete time delay.

3.2.3.1. Scalar Multiplication

The question is how to represent graphically a discrete signal multiplied by a scalar C. This is done according to the following illustration (**Fig.3.10**):



We can also do this representation as:

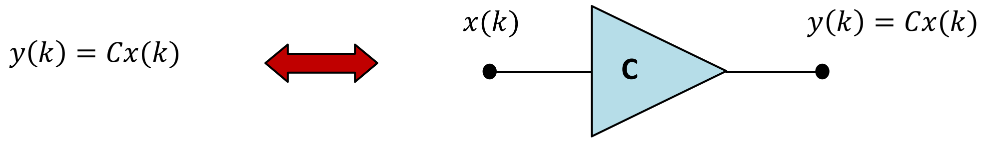


Fig. 3.10 Block diagram representation of scalar multiplication operator

3.2.3.2. Algebraic Sum

The algebraic sum allows us to represent graphically several signals when they are summed up to produce an output signal.

If $x_1(k)$, $x_2(k)$ and $x_3(k)$ are supposed to be three signals scaled respectively by the gains C_1 , C_2 and C_3 which are applied at the input of an LTI discrete system to produce the output signal $y(k)$. The block diagram representation of this sum of signals is shown as in **Fig.3.11**:

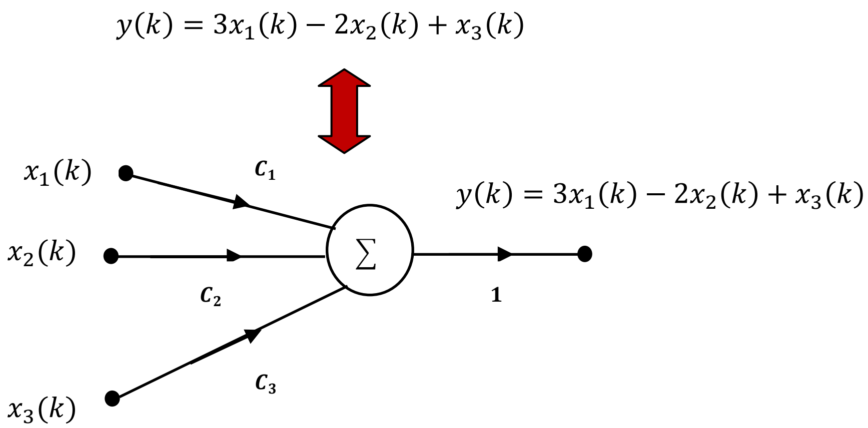


Fig. 3.11 Block diagram representation of algebraic sum

3.2.3.3 Discrete Time Delay

In discrete time systems, time delay is represented by the operator z^{-1}. The block diagram representation of the general case of time delay is given in **Fig.3.12**.

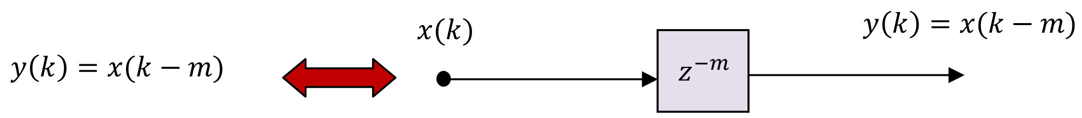


Fig. 3.12 Block diagram representation of discrete time delay

Illustrative example:

Using block diagram representation, a typical LTI discrete (sampled data) system is shown in **Fig.3.13**.

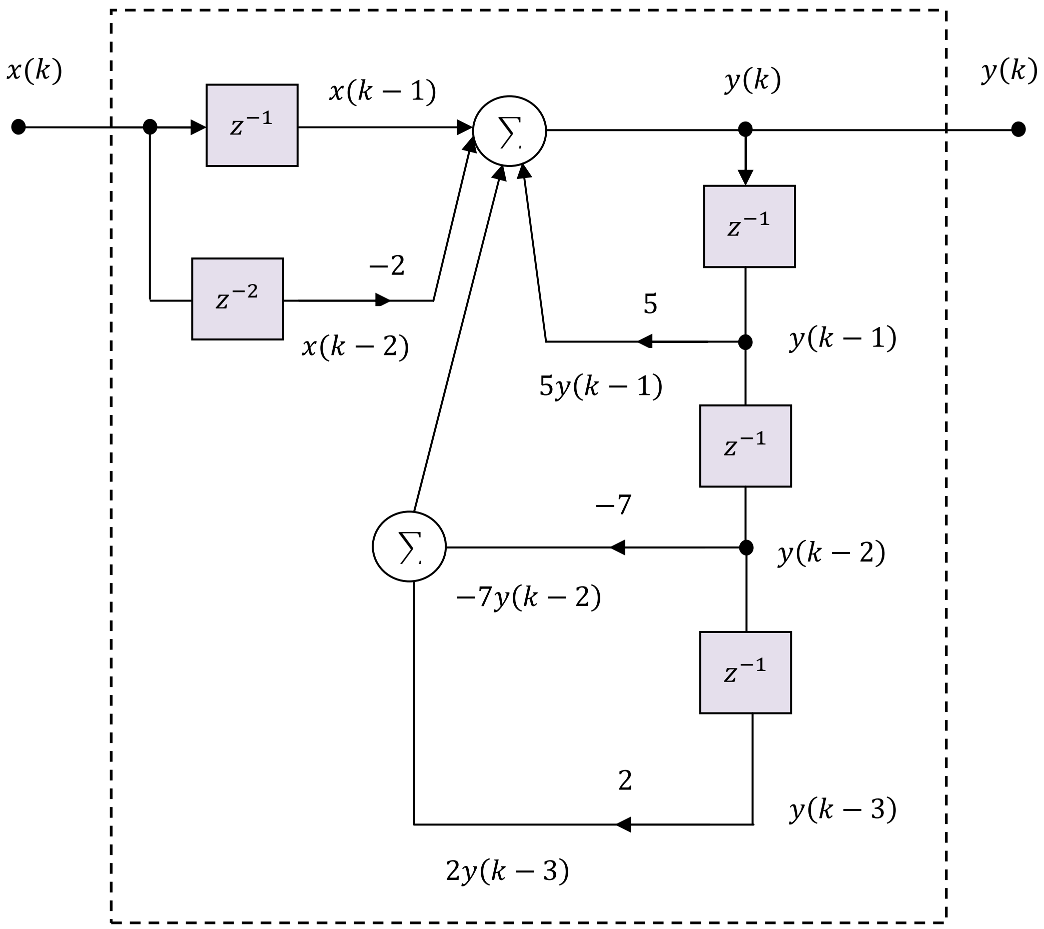


Fig.3. 13 Typical block diagram representation of LTI discrete system

2.3.4. From block diagram to discrete transfer function representation

We can obtain the discrete transfer function representing the behavior of any discrete system given its block diagram representation. The discrete transfer function is generated by passing through the difference equation determination from the block diagram. In the following illustrative example, we show the step-by-step procedure of so doing.

Example:

Consider the discrete (digital) system represented by the block diagram shown in **Fig.3.13; d**erive the corresponding discrete transfer function of the system?

Solution:

From the block diagram representation of **Fig.3.13,** we can easily write the corresponding difference equation describing the behavior of the digital system as:

$$y(k) = x(k-1) - 2x(k-2) + 5y(k-1) - 7y(k-2) + 2y(k-3) \qquad (3.13)$$

By applying Z transform on both sides of eq.(3.13), we get:

$$Z\{y(k)\} = Z\{x(k-1) - 2x(k-2) + 5y(k-1) - 7y(k-2) + 2y(k-3)\}$$

Using Z transform properties particularly that of linearity and time delay, it results:

$$Y(z) = z^{-1}X(z) - 2z^{-2}X(z) + 5z^{-1}Y(z) - 7z^{-2}Y(z) + 2z^{-3}Y(z)$$

We arrange the input samples in one side and the output samples in one side, it becomes:

$$Y(z) - 5z^{-1}Y(z) + 7z^{-2}Y(z) - 2z^{-3}Y(z) = z^{-1}X(z) - 2z^{-2}X(z)$$

$$\Rightarrow (1 - 5z^{-1} + 7z^{-2} - 2z^{-3})Y(z) = (z^{-1} - 2z^{-2})X(z)$$

We apply the definition of discrete transfer function, we write:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 2z^{-2}}{1 - 5z^{-1} + 7z^{-2} - 2z^{-3}}$$

We can express $G(z)$ under the form of ratio of polynomials as function of z as:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 2z^{-2}}{1 - 5z^{-1} + 7z^{-2} - 2z^{-3}} \times \frac{z^3}{z^3} = \frac{z^2 - 2z}{z^3 - 5z^2 + 7z - 2}$$

Consequently, we have explained the relationship between the different representations of any given discrete or digital system. we have also mention how to move from one representation to another by just applying either Z transform or its inverse with their appropriate properties.

4. Discretizing LTI continuous time systems

We have seen in the first chapter the use of sampler to convert a continuous time (analog) signal into discrete time (digital) signal. The operation was called sampling or discretization and the electronic device performing this conversion is the analog to digital converter (ADC) which is modeled by an ideal switch as long as the analysis and design of digital control system is concerned.

On the other hand, the design and analysis of sampled data control system often requires the discretization of a continuous time system. Usually this continuous time system to be discretized is represented by the Laplace transfer function.

In this section we explore the different methods widely used and employed to obtain discrete transfer function from the existing Laplace transfer function, hence discrete system representation is obtained from that of continuous system. In this vein, the following discretizing methods can be distinguished.

(1) Discretization using **Zero Order Hold** (ZOH) method.

(2) Discretization using **One Order Hold** (1OH) method.

(3) Discretization using **Tustin approximation** method.

(4) Discretization using **Euler approximation** method.

4.1. Discretization using Zero Order Hold (ZOH) method

The discretization of a continuous time system represented by an s-transfer function using the method of zero order hold (ZOH) is done by preceding the continuous transfer function $G(s)$ with the ZOH block (transfer function) $G_{ZOH}(s)$. This is illustrated by the following block diagram of **Fig.3.14**.

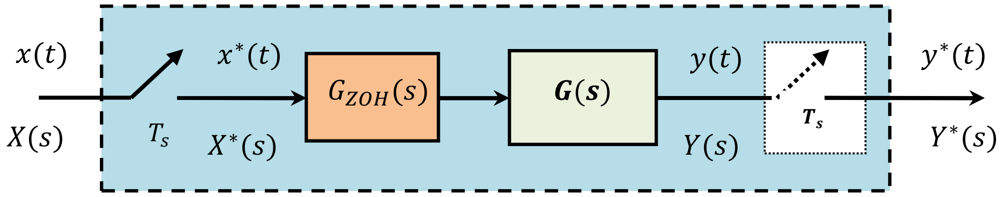


Fig.3. 14 Block diagram representation of continuous system discretization using ZOH method

The discrete LTI system obtained using ZOH discretization method is represented by the following block diagram (**Fig.3.15**).

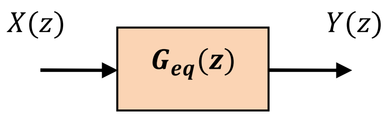


Fig.3. 15 Block diagram of the equivalent discretized LTI system

With:

$G_{eq}(z)$: is the equivalent z transfer function of the resulted discrete system and is calculated as:

$$G_{eq}(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right] \tag{3.14}$$

4.2. Discretization using One Order Hold (1OH) method

The one order hold (1OH) can also be used as a discretization method to obtain a discrete (digital) system from a continuous system. It is similar to ZOH method with the use of 1OH block instead of ZOH block. Hence, using this method, the block diagram illustration is depicted in **Fig.3.16**:

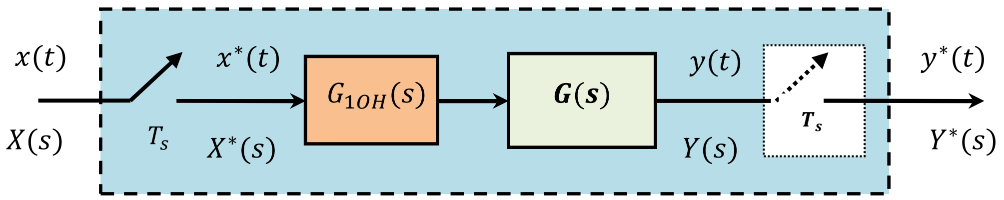


Fig.3. 16 Block diagram representation of continuous system discretization using 1OH method

The discrete LTI system obtained using 1OH discretization method is represented by the following block diagram of **Fig.3.17**.

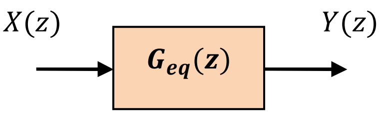


Fig.3. 17 Block diagram of the equivalent discretized LTI system using 1OH method

With:

T_s: is the sampling period.

$G_{eq}(z)$: is the equivalent z transfer function of the resulted discrete system and is calculated as:

$$G_{eq}(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1})^2 \mathcal{Z}\left[\frac{1 + T_s\, s}{T_s\, s^2} G(s)\right] \tag{3.15}$$

4.3. Discretization using Tustin approximation method

The Tustin approximation, also known as Tustin Bilinear approximation, is used to convert a continuous time system into discrete time system. The equivalent z transfer function of resulted discrete system is obtained by performing in the s transfer function the following direct replacement:

$$s = \frac{2}{T_s}\frac{(z-1)}{(z+1)} \tag{3.16}$$

With :

T_s: is the sampling period.

Hence, the equivalent z transfer function representing the discrete system is generated and written:

$$G_{eq}(z) = \left. G(s)\right|_{s=\frac{2\,(z-1)}{T_s(z+1)}} \tag{3.17}$$

We can illustrate Tustin approximation method of discretizing a continuous time system by the following block diagram of **Fig.3.18**.

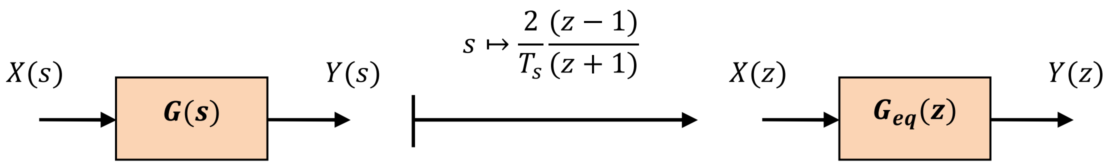


Fig.3. 18 Block diagram representation of continuous system discretization using **Tustin** method

4.4. Discretization using Euler approximation method

The Euler's method of discretizing a continuous time system relies on Euler's principle of approximating the first derivative of a continuous function between two consecutive sampling instants. In fact, Euler's method contains two approximate versions known respectively as forward and backward approximations.

4.4.1. Discretization using Euler Backward approximation method

Euler's backward approximation used to discretize a continuous system as it is represented by an s-transfer function consists of direct replacing the complex variable 's' by: $\frac{(z-1)}{T_s z}$. The illustration of this approximation is depicted in **Fig.3.19**.

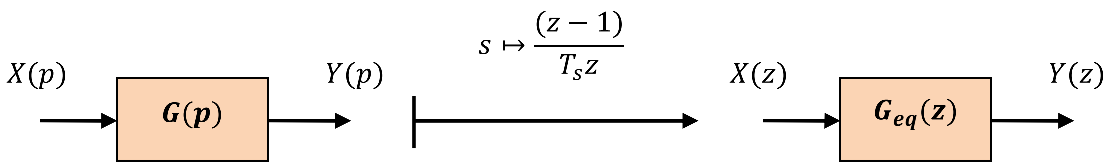


Fig.3. 19 Block diagram representation of continuous system discretization using **Backward Euler's** method

Using this method, the resulting equivalent z (discrete) transfer function can be obtained as:

$$G_{eq}(z) = G(s)\Big|_{s=\frac{(z-1)}{T_s z}} \tag{3.18}$$

Of course, in all the mentioned approximations, T_s is the sampling period.

4.4.2. Discretization using Euler Forward approximation method

The Forward version of Euler's approximation method for discretization is based on replacing the complex variable 's' by: $\frac{(z-1)}{z}$. Therefore, using this approximation, the equivalent discrete transfer function obtained from the s transfer function is generated as:

$$G_{eq}(z) = G(s)|_{s=\frac{(z-1)}{z}} \tag{3.19}$$

The block diagram representation of the discretized system using Euler's Forward approximation is illustrated in the following figure (**Fig.3.20**).

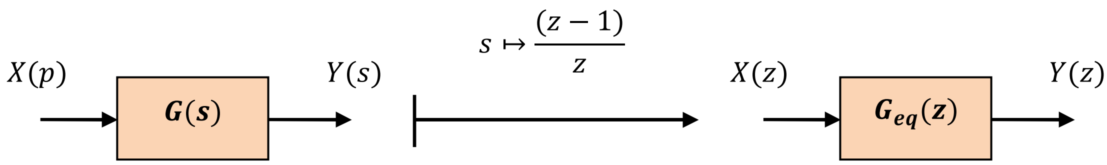


Fig.3. 20 Block diagram representation of continuous system discretization using **Forward Euler's** method

Example:

At the end of this section, we consider this example of discretizing a continuous time system modeled by the Laplace transfer function.

Assume a continuous time system given by the s transfer function as:

$$G(s) = \frac{2}{(3+5s)}.$$

Q: / using the aforementioned discretization methods, obtain the equivalent and corresponding z transfer function $G(z)$?

Answer:

1) Using Zero Order Hold (ZOH)

Using (3.13), the equivalent z transfer function is:

$$G(z) = \frac{Y(z)}{X(z)} = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{2/5}{s\left(s+\frac{3}{5}\right)}\right]$$

Applying partial fraction expansion on $\frac{2/5}{s\left(s+\frac{3}{5}\right)}$, we can write :

$$\frac{2/5}{s\left(s+\frac{3}{5}\right)}=\frac{A_1}{s}+\frac{A_2}{s+\frac{3}{5}}$$

With:

$$A_1=s.\frac{2/5}{s\left(s+\frac{3}{5}\right)}\Bigg|_{s=0}=\frac{2/5}{\left(s+\frac{3}{5}\right)}\Bigg|_{s=0}=\frac{2}{3}$$

$$A_2=\left(s+\frac{3}{5}\right).\frac{2/5}{s\left(s+\frac{3}{5}\right)}\Bigg|_{s=0}=\frac{2/5}{s}\Bigg|_{s=-\frac{3}{5}}=-\frac{2}{3}$$

That is :

$$\frac{2/5}{s\left(s+\frac{3}{5}\right)}=\frac{\frac{2}{3}}{s}-\frac{\frac{2}{3}}{s+\frac{3}{5}}$$

It results:

$$\mathcal{Z}\left[\frac{2/5}{s\left(s+\frac{3}{5}\right)}\right]=\mathcal{Z}\left[\frac{\frac{2}{3}}{s}-\frac{\frac{2}{3}}{s+\frac{3}{5}}\right]=\frac{2}{3}\mathcal{Z}\left[\frac{1}{s}\right]-\frac{2}{3}\mathcal{Z}\left[\frac{1}{s+\frac{3}{5}}\right]$$

By using the correspondence table between Laplace transform and Z transform, we get:

$$\mathcal{Z}\left[\frac{2/5}{s\left(s+\frac{3}{5}\right)}\right]=\frac{2}{3}\frac{z}{z-1}-\frac{2}{3}\frac{z}{z-e^{\left(-\frac{3}{5}T_s\right)}}$$

Taking: $T_s=1\ sec$, then :

$$\mathcal{Z}\left[\frac{2/5}{s\left(s+\frac{3}{5}\right)}\right]=\frac{2}{3}\frac{z}{z-1}-\frac{2}{3}\frac{z}{z-e^{\left(-\frac{3}{5}\right)}}=\frac{2}{3}\frac{z}{z-1}-\frac{2}{3}\frac{z}{z-0.5488}$$

Finally:

$$G(z) = (1 - z^{-1})\left[\frac{2}{3}\frac{z}{z-1} - \frac{2}{3}\frac{z}{z-0.5488}\right] = \frac{z-1}{z}\left[\frac{2}{3}\frac{z}{z-1} - \frac{2}{3}\frac{z}{z-0.5488}\right]$$

$$G(z) = \frac{2}{3} - \frac{2}{3}\left(\frac{z-1}{z-0.5488}\right) = \frac{2}{3}\left(\frac{z-0.5488-z+1}{z-0.5488}\right) = \frac{2}{3}\left(\frac{0.4512}{z-0.5488}\right)$$

2) Using One Order Hold (OOH) Discretization Method

Using this method, the equivalent z transfer function of the given s transfer function is obtained as follows:

$$G(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1})^2 \mathcal{Z}\left[\frac{1 + T_s\, s}{T_s\, s^2} G(s)\right]$$

Taking: $T_s = 1\ sec$, then :

$$G(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1})^2 \mathcal{Z}\left[\frac{1 + s}{s^2} G(s)\right] = \frac{2}{5}(1 - z^{-1})^2 \mathcal{Z}\left[\frac{(1 + s)}{s^2\left(s + \frac{3}{5}\right)}\right]$$

Applying partial fraction expansion on the s transfer function: $\frac{(1+s)}{s^2\left(s+\frac{3}{5}\right)}$, we have:

$$\frac{(1 + s)}{s^2\left(s + \frac{3}{5}\right)} = \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{A_3}{s + \frac{3}{5}}$$

Where the coefficients are determined as:

$$A_{11} = \frac{1}{0!}\frac{d^{(0)}}{ds^{(0)}}\left[s^2 \frac{(1 + s)}{s^2\left(s + \frac{3}{5}\right)}\right]\Bigg|_{s=0} = \left[\frac{(1 + s)}{\left(s + \frac{3}{5}\right)}\right]\Bigg|_{s=0} = \frac{5}{3}$$

$$A_{12} = \frac{1}{1!}\frac{d^{(1)}}{ds^{(1)}}\left[s^2 \frac{(1 + s)}{s^2\left(s + \frac{3}{5}\right)}\right]\Bigg|_{s=0} = \frac{d^{(1)}}{ds^{(1)}}\left[\frac{(1 + s)}{\left(s + \frac{3}{5}\right)}\right]\Bigg|_{s=0}$$

$$= \left[\frac{(1 + s) - \left(s + \frac{3}{5}\right)}{\left(s + \frac{3}{5}\right)^2}\right]\Bigg|_{s=0} = \frac{2}{3}$$

$$A_3 = \left[\left(s+\frac{3}{5}\right)\frac{(1+\ s)}{s^2\left(s+\frac{3}{5}\right)}\right]\Bigg|_{s=-\frac{3}{5}} = \left[\frac{(1+\ s)}{s^2}\right]\Bigg|_{s=-\frac{3}{5}} = \frac{10}{9}$$

Hence:

$$\frac{(1+\ s)}{s^2\left(s+\frac{3}{5}\right)} = \frac{\frac{5}{3}}{s^2} + \frac{\frac{2}{3}}{s} + \frac{\frac{10}{9}}{s+\frac{3}{5}}$$

It results after using Z transform properties :

$$\mathcal{Z}\left[\frac{(1+\ s)}{s^2\left(s+\frac{3}{5}\right)}\right] = \mathcal{Z}\left[\frac{\frac{5}{3}}{s^2} + \frac{\frac{2}{3}}{s} + \frac{\frac{10}{9}}{s+\frac{3}{5}}\right] = \frac{5}{3}\mathcal{Z}\left[\frac{1}{s^2}\right] + \frac{2}{3}\mathcal{Z}\left[\frac{1}{s}\right] + \frac{10}{9}\mathcal{Z}\left[\frac{1}{s+\frac{3}{5}}\right]$$

Taking $T_s = 1\ sec$ and using the table of correspondence between z and s transforms, we get:

$$\mathcal{Z}\left[\frac{(1+\ s)}{s^2\left(s+\frac{3}{5}\right)}\right] = \frac{5}{3}\frac{z}{(z-1)^2} + \frac{2}{3}\frac{z}{(z-1)} + \frac{10}{9}\frac{z}{(z-0.5488)}$$

Finally :

$$G(z) = \frac{2}{5}(1-z^{-1})^2\mathcal{Z}\left[\frac{(1+\ s)}{s^2\left(s+\frac{3}{5}\right)}\right]$$

$$= \frac{2}{5}(1-z^{-1})^2\left[\frac{5}{3}\frac{z}{(z-1)^2} + \frac{2}{3}\frac{z}{(z-1)} + \frac{10}{9}\frac{z}{(z-0.5488)}\right]$$

$$G(z) = \frac{2}{5}\frac{(z-1)^2}{z^2}\left[\frac{5}{3}\frac{z}{(z-1)^2} + \frac{2}{3}\frac{z}{(z-1)} + \frac{10}{9}\frac{z}{(z-0.5488)}\right]$$

$$G(z) = \left[\frac{2}{3}\frac{1}{z} + \frac{4}{15}\frac{(z-1)}{z} + \frac{4}{9}\frac{(z-1)^2}{z(z-0.5488)}\right]$$

$$= \frac{2}{45}\left[\frac{3(z-0.5488)(3+2z)+10(z-1)^2}{z(z-0.5488)}\right]$$

3) Using « Tustin » approximation Method :

By direct use of the formula (3.15), and taking $T_s = 1\,sec$ for simplicity, the equivalent discrete transfer function is:

$$G_{eq}(z) = G(s)|_{s=\frac{2\,(z-1)}{T_s(z+1)}} = \frac{2}{\left(3+10\frac{(z-1)}{(z+1)}\right)} = \frac{2}{\frac{3(z+1)+10(z-1)}{(z+1)}}$$

$$G_{eq}(z) = \frac{2(z+1)}{13z-7}$$

4) Using Euler's Backward Method

From the formula (3.16), the discrete transfer function using this method version of Euler is:

$$G_{eq}(z) = G(s)|_{s=\frac{(z-1)}{T_S z}} = \frac{2}{\left(3+5\frac{(z-1)}{z}\right)} = \frac{2}{\frac{3z+5z-5}{z}} = \frac{2z}{8z-5}$$

5) Using Euler's Forward Method

From the formula (3.16), the discrete transfer function using this method version of Euler is:

$$G_{eq}(z) = G(s)|_{s=\frac{(z-1)}{z}} = \frac{2}{\left(3+5\frac{(z-1)}{z}\right)} = \frac{2}{\frac{3z+5z-5}{z}} = \frac{2z}{8z-5}$$

This reveals that when the sampling period equals 1, the **Backward** and **Forward** versions of Euler are identical.

5. Equivalent transfer function of complex discrete control system

In this section, we consider the calculation and determination of the equivalent and global discrete transfer function of a sampled data control system consisted of an interconnection of elementary sub-systems. Any digital control system is found under two main structures depending on the way the elementary sub-systems are connected. These two structures are:

- **Open loop discrete control system**: when the sub-systems are connected in series (or in cascade).

- **Closed loop discrete control system**: when the sub-systems are connected in both cascade and parallel structure.

In developing the content of this subject, we will discover that the final expression of the equivalent discrete transfer function of the sampled data control system depends on the following two factors:

(1) The number of the samplers used in the control system block diagram.

(2) The position of the samplers regarding the elementary sub-systems.

5.1. Case of Open Loop discrete control system

In order to well understanding how to determine the equivalent discrete transfer function for given open loop discrete control system, we distinguish the following cases of how both the subsystems and the samplers are interconnected.

5.1.1 Case of the continuous time system is between two samplers

This situation can be depicted as in **Fig. 3. 21**.

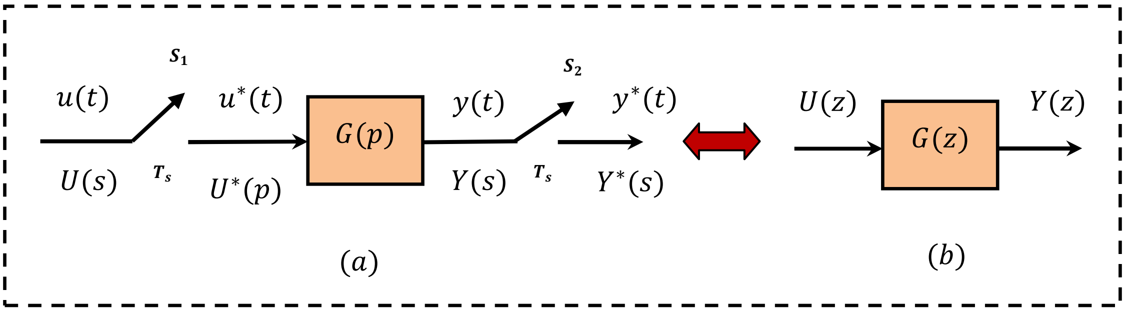


Fig.3. 21: **(a)** Block diagram of continuous time open loop system surrounded by two samplers, **(b)** Block diagram of the equivalent discrete system

In this case, the equivalent transfer function of the discrete system is determined according to the following procedure:

- Before the sampler S_1, we have: $u(t) \mapsto \overset{L}{\rightarrow} U(s)$
- After the sampler S_1, we have: $u(t) \mapsto \overset{S_1}{\rightarrow} u^*(t) \quad \Rightarrow U(s) \mapsto \overset{S_1}{\rightarrow} U^*(s)$
- Before the sampler S_2, we have: $y(t) \mapsto \overset{L}{\rightarrow} Y(s) = U^*(s).G(s)$
- After the sampler S_2, we have: $y(t) \mapsto \overset{S_2}{\rightarrow} y^*(t) \quad \Rightarrow Y(s) \mapsto \overset{S_2}{\rightarrow} Y^*(s)$

Where L denotes the Laplace transform operator.

Using the operation principles applied on the given block diagram of **Fig.3. 21 (a)**, we can perform the following:

$$Y^*(s) = [Y(s)]^* = [U^*(s).G(s)]^* = U^*(s)[G(s)]^* = U^*(s)G^*(s)$$

$$\Rightarrow \begin{cases} Y^*(s) \mapsto \xrightarrow{z=e^{s.T_S}} Y(z) \\ U^*(s) \mapsto \xrightarrow{z=e^{s.T_S}} U(z) \\ G^*(s) \mapsto \xrightarrow{z=e^{s.T_S}} G(z) \end{cases}$$

From which, we obtain :

$$Y(z) = U(z)G(z)$$

Consequently :

$$Y(z) = U(z)G(z) \Rightarrow G(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}[Y(s)]}{\mathcal{Z}[U(s)]} = \mathcal{Z}[G(s)]$$

Hence, we can state and apply the following rule which is used to determine the equivalent discrete transfer function corresponding to a continuous time system situated between two samplers.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}[Y(s)]}{\mathcal{Z}[U(s)]} = \mathcal{Z}[G(s)] \tag{3.20}$$

With $\mathcal{Z}$ denotes the z transform operator.

5.1.2 Case of two cascaded continuous time systems with an intermediate sampler

This case is illustrated by the following **Fig.3. 22**.

To determine the equivalent discrete transfer function for this case, we can follow similar steps as in the previous case and we arrive to the following result:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}[Y(p)]}{\mathcal{Z}[U(p)]} = \mathcal{Z}[G_1(s)]\mathcal{Z}[G_2(s)] \tag{3.21}$$

With $\mathcal{Z}$ denotes the z transform operator.

Expression (3.21) can also be written as:

$$G(z) = G_1(z)G_2(z) \tag{3.22}$$

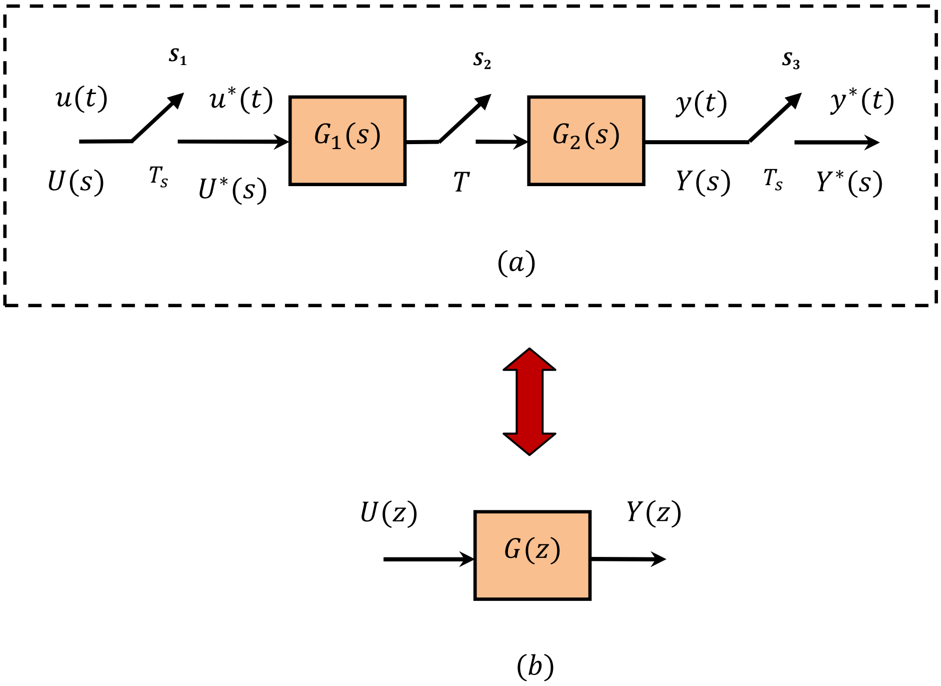


Fig.3. 22: **(a)** Block diagram of two cascade continuous time system with an intermediate sampler
(b) Block diagram of the equivalent discrete system

5.1.3 Case of two cascaded continuous systems without an intermediate sampler

This can be illustrated as in **Fig.3.23**

By applying a similar procedure, we determine the equivalent discrete transfer function for this case and we end up to the following result:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}[Y(s)]}{\mathcal{Z}[U(s)]} = \mathcal{Z}[G_1(s)G_2(s)] \qquad (2.23)$$

With $\mathcal{Z}$ denotes the z transform operator.

Expression (3.23) can also be written as:

$$G(z) = \frac{Y(z)}{U(z)} = [G_1G_2](z) \qquad (3.24)$$

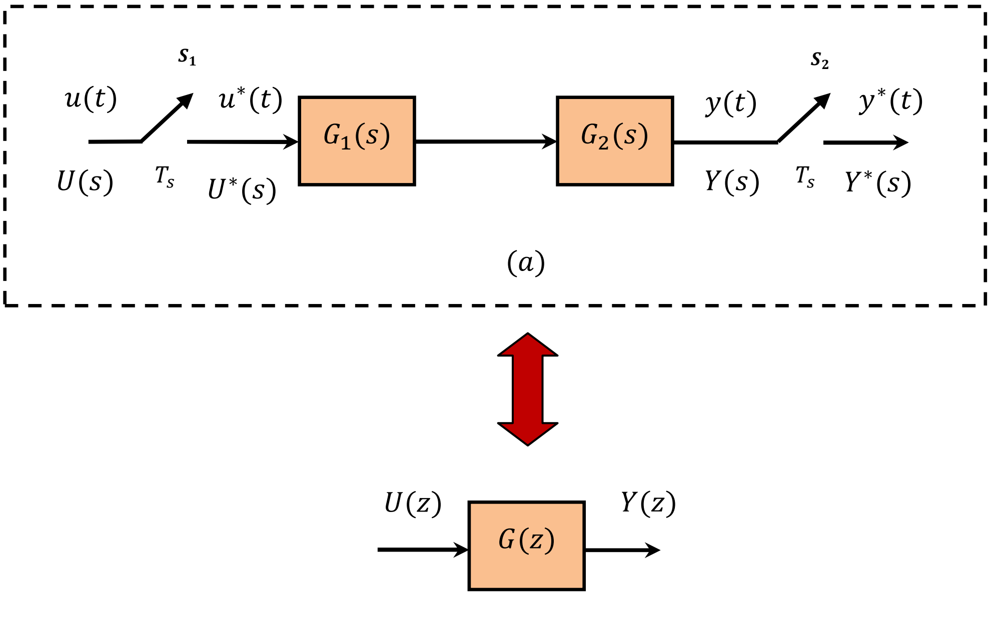


Fig.3. 23: (a) Block diagram of two cascade continuous time system without an intermediate sampler
(b) Block diagram of the equivalent discrete system

5.2. Case of Closed Loop discrete control system

In fact the determination of the equivalent discrete transfer function of a closed loop discrete (sampled data) control system is not unique even for the same structure and topology, because this fundamentally depends on both the number of samplers used and their locations within the system's structure. For the sake of illustration, we will work out a typical closed loop discrete control system represented by the following block diagram of **Fig.3.24**.

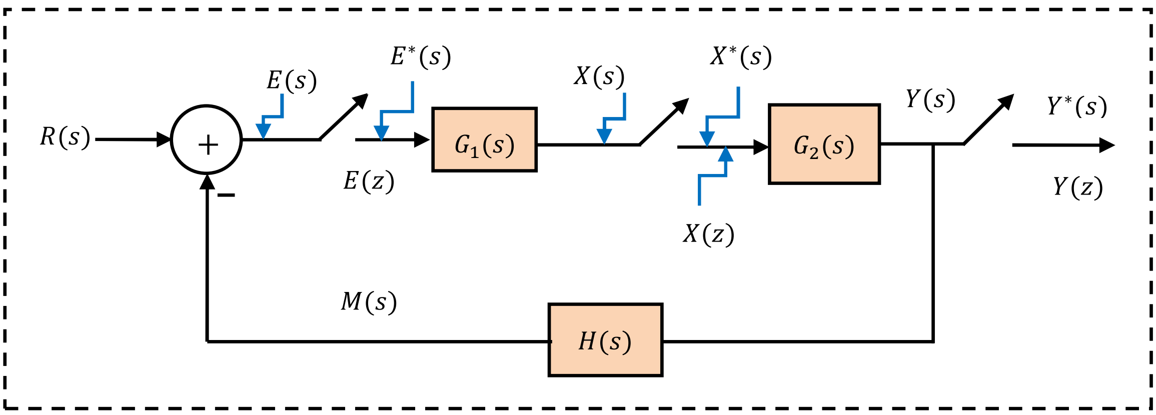


Fig.3. 24: Block diagram of typical sampled data closed loop control system

To determine the equivalent discrete transfer function of the closed loop sampled data control system described by the block diagram of **Fig.3.24**, we proceed as follows:

We have:

$$E^*(s) = [E(s)]^* = [R(s) - M(s)]^* = [R(s) - H(s)Y(s)]^* \tag{3.25}$$

By applying a virtual sampler at the input, we get:

$$[R(s) - H(s)Y(s)]^* = R^*(s) - [H(s)Y(s)]^* \tag{3.26}$$

From (3.18) and (3.19), we can write:

$$E^*(s) = [E(s)]^* = R^*(s) - [H(s)G_2(s){G_1}^*(s)E^*(s)]^* \tag{3.27}$$

At the sampling time instant, (3.27) gives:

$$E^*(s) = [E(s)]^* = R^*(s) - [H(s)G_2(s)]^*{G_1}^*(s)E^*(s) \tag{3.28}$$

From which we can write:

$$\{1 + [H(s)G_2(s)]^*{G_1}^*(s)\}E^*(s) = R^*(s) \tag{3.29}$$

On the other hand:

$$Y^*(s) = [Y(s)]^* = [G_2(s)X^*(s)]^* = {G_2}^*(s)X^*(s) = {G_2}^*(s){G_1}^*(s)E^*(s) \tag{3.30}$$

From (3.30), we can write:

$$E^*(s) = \frac{Y^*(s)}{{G_2}^*(s){G_1}^*(s)} \tag{3.31}$$

By substituting (3.31) into (3.29), we obtain:

$$\{1 + [H(s)G_2(s)]^*{G_1}^*(s)\}\frac{Y^*(s)}{{G_2}^*(s){G_1}^*(s)} = R^*(s) \tag{3.32}$$

After some manipulating steps and simplification, we end up to the following result:

$$\frac{Y^*(s)}{R^*(s)} = \frac{{G_2}^*(s){G_1}^*(s)}{1 + [H(s)G_2(s)]^*{G_1}^*(s)}R^*(s) \tag{3.33}$$

Considering the sampling time instant, we apply Z transform on both sides of **(3.33)** to get:

$$Z\left\{\frac{Y^*(s)}{R^*(s)}\right\} = Z\{G^*(s)\} = Z\left\{\frac{{G_2}^*(s){G_1}^*(s)}{1+[H(s)G_2(s)]^*{G_1}^*(s)}R^*(s)\right\} \tag{3.34}$$

Using the previous notions on Z transform application, the following final result represents the determined equivalent discrete transfer function that corresponds to the typical closed loop sampled data control system described by the block diagram of **Fig.3.24**.

$$G(z) = \frac{Y(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1+[HG_2](z)G_1(z)} \tag{3.35}$$

Where:

$$\begin{cases} G(z) = Z\{G^*(s)\} \\ Y(z) = Z\{Y^*(s)\} \\ R(z) = Z\{R^*(s)\} \\ G_1(z)G_2(z) = Z\{{G_1}^*(s){G_2}^*(s)\} \\ G_1(z) = Z\{{G_1}^*(s)\} \\ [HG_2](z) = Z\{[H(s)G_2(s)]^*\} \end{cases} \tag{3.36}$$

The above development and analysis can be illustrated as it is shown in **Fig.3.25**

Based on the aforementioned demonstration and development used to determine the equivalent discrete transfer function of the typical closed loop sampled data control system, and due to the fact that this transfer function changes from one structure to another depending on the number of samplers and their locations within the system, the following closed loop sampled data control systems are further given and can be encountered. The reader can apply the same procedure to determine their corresponding equivalent discrete transfer functions.

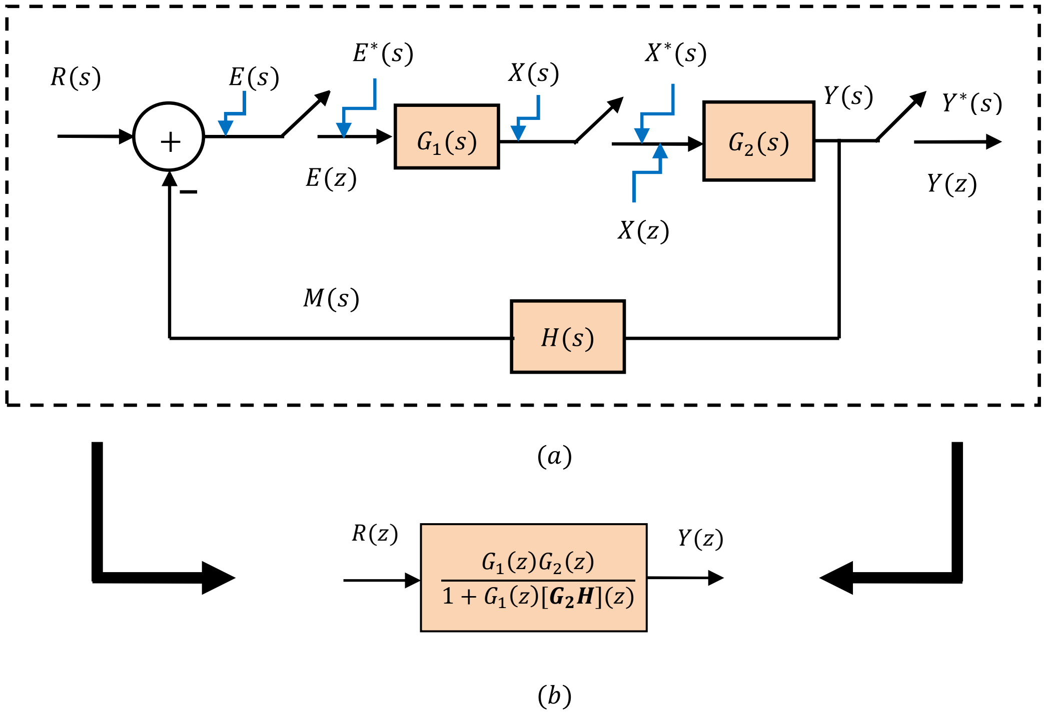
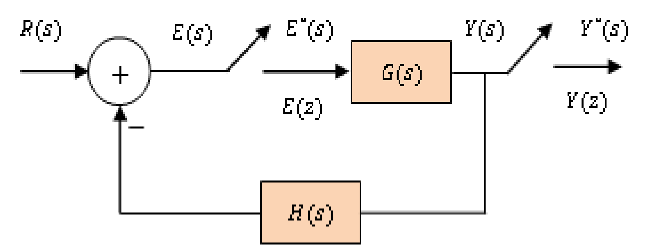
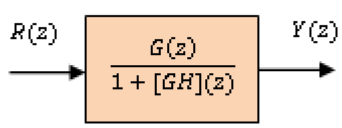
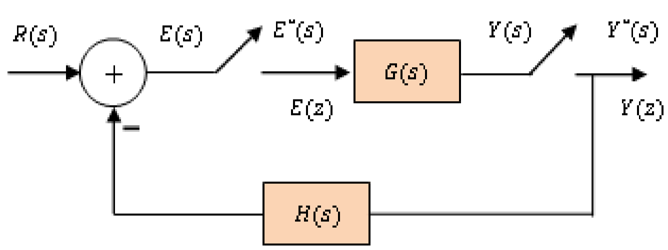
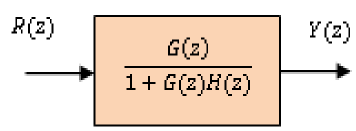
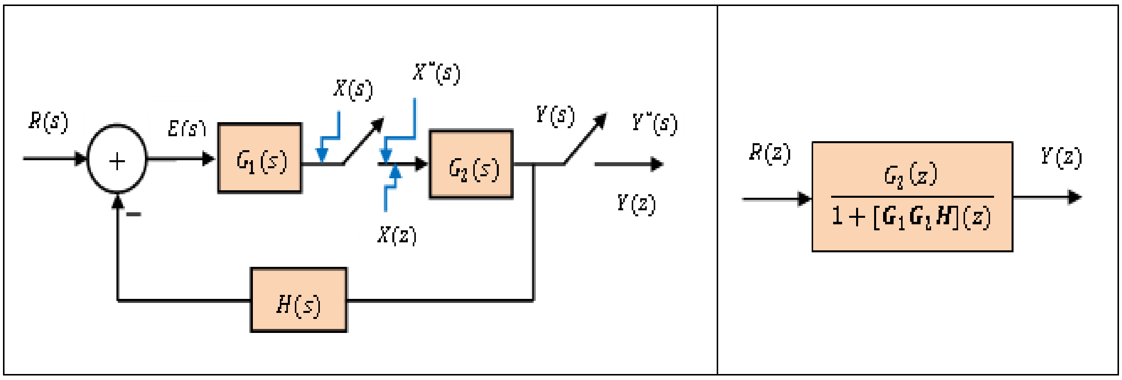


Fig.3.25: (a) Block diagram of typical sampled data closed loop control system

(b) Block diagram of the equivalent discrete control system

Table 3.1 Possible closed loop structures of sampled data control system and their Equivalent discrete transfer functions

Possible closed loop structure of sampled data control system	**Equivalent discrete transfer function**
	
	



6. Equivalent transfer function of Associated LTI discrete systems

Linear Time Invariant (LTI) discrete systems represented by their corresponding discrete transfer functions can be associated using three topologies which are:

- Serial structure association.
- Parallel structure association.
- Feedback structure association.

In this section, we will show how to determine the equivalent discrete transfer function of discrete systems when connected under these structures.

6.1. Case of two discrete systems associated in series

The equivalent discrete transfer function of two discrete systems connected in series (also called two systems in cascade) is determined as it is illustrated in the following **Fig.3.26**.

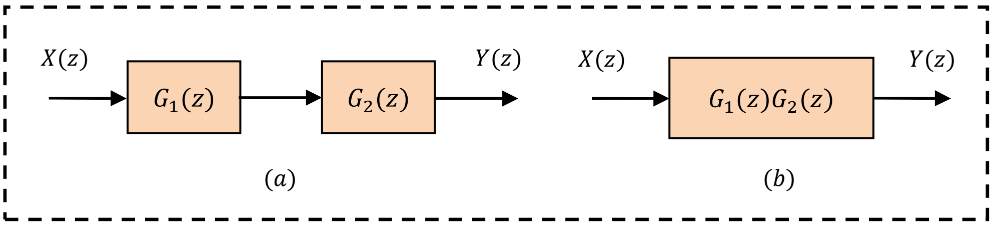


Fig.3.26 Association of two discrete systems in series, **(a)** original block diagram, **(b)** The equivalent block diagram and the corresponding discrete transfer function

6.2. Case of two discrete systems associated in parallel

In the following figure (**Fig.3.27**) we illustrate how the equivalent discrete transfer function corresponding to two discrete systems connected in parallel is determined.

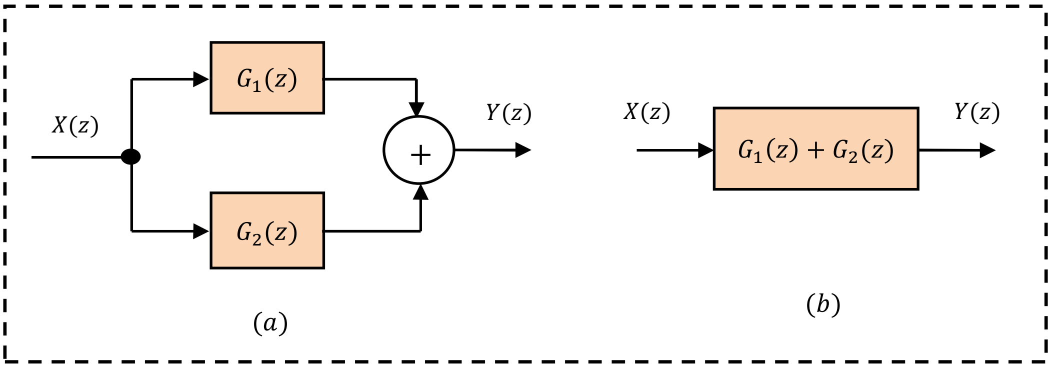


Fig.3.27 Association of two discrete systems in parallel, **(a)** original block diagram, **(b)** The equivalent block diagram and the corresponding discrete transfer function

6.3. Case of two discrete systems associated in feedback structure

When two discrete systems are associated in a feedback structure, the equivalent block diagram with the corresponding equivalent discrete transfer function is determined as it is shown and illustrated in the following **Fig.3.28**.

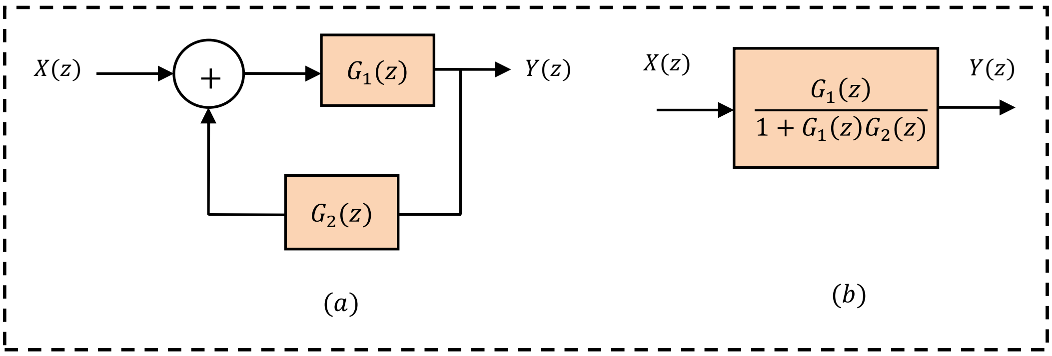


Fig.3.28 Association of two discrete systems in feedback structure, **(a)** original block diagram, **(b)** the equivalent block diagram and the corresponding discrete transfer function