Chapter 2: Z-Transform of sampled signals

1. Introduction

As it was stated in the previous chapter that a signal is a physical quantity that contains and carries data and information. Similarly, a system is a mechanism that establishes and responds to a relationship between the different affecting input signals. These signals need to be processed in order to study and analyze the system's behavior. As it is the case of continuous time systems, the Laplace transform represents the powerful mathematical tool that allows these study and analysis, in discrete time case, Z transform is instead used.

2. Mathematical Definition of Z Transform

If we assume $f^*(t) = f(kT_s) = f(k)$ to be a given sampled (discrete time) signal, its Z transform denoted as $F(z)$ is defined as a function of the complex variable 'z' by the following mathematical expression:

$$F(z) = \mathcal{Z}\,\{f(kT_s)\} = \mathcal{Z}\,\{f^*(t)\} = \sum_{k=\infty}^{\infty} f(kT_s)z^{-k} \qquad (2.1)$$

The expression (2.1) defines what is called two sided Z transform for any discrete time signal. As a particular case which concerns the causal discrete signals and systems, the one sided Z transform is defined as:

$$F(z) = \mathcal{Z}\,\{f(kT_s)\} = \mathcal{Z}\,\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT_s)z^{-k} \qquad (2.2)$$

In both definitions, the symbol $\mathcal{Z}$ is used to denote the Z transform operator.

3. Derivation of Z transform

Before tackling the mathematical derivation of Z transform expression, we introduce the following definition of Laplace transform of discrete time (sampled) signal.

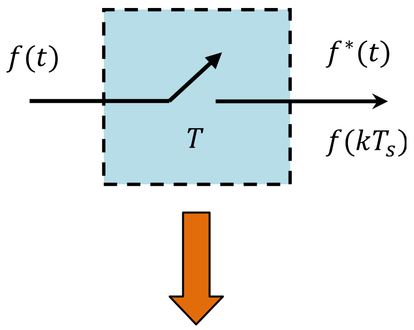
3.1. Laplace Transform of sampled signal

We consider $f^*(t)$ being the sampled signal of the continuous time signal $f(t)$, we accept without proof the definition of Laplace transform of the signal $f^*(t)$ denoted by $F^*(s)$ and given by:

$$F^*(s) = L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT_s)e^{-skT_s} \tag{2.3}$$

$F^*(s)$: is also called stared Laplace transform.

In the following figure (**Fig.2.1**), we give an illustration of how the Laplace transform $F^*(s)$ is calculated:



$$f^*(t) = \sum_{k=0}^{\infty} f(kT_s)\delta(t-kT_s) \xrightarrow{L} F^*(s) = \sum_{k=0}^{\infty} f(kT_s)e^{-skT_s}$$

Fig. 2.1 Illustration showing the calculation of stared Laplace transform $F^*(s)$

After we have introduced this concept of Laplace transform of sampled signal, we are at the level of deriving the expression defining the Z transform.

Let us consider $f^*(t)$ being the causal sampled signal expressed as:

$$f^*(t) = f(kT_s) = f(t).\delta_{T_s}(t) = f(t).\sum_{k=0}^{\infty}\delta(t-kT_s)$$

$$f^*(t) = f(kT_s) = \sum_{k=0}^{\infty} f(kT_s).\delta(t-kT_s) \tag{2.4}$$

By applying stared Laplace transform given by (2.3) on expression (2.4), we get:

$$F^*(s) = L\{f^*(t)\} = \sum_{k=0}^{\infty}\left[\sum_{k=0}^{\infty} f(kT_s).\delta(t-kT_s))\right]e^{-skT_s}$$

$$F^*(s) = \sum_{k=0}^{\infty} f(kT_s)e^{-skT_s} \qquad (2.5)$$

Then, we use the variable transformation given by:

$$z = e^{sT_s} \qquad (2.6)$$

The equation (2.5) becomes:

$$F^*(s) = \sum_{k=0}^{\infty} f(kT_s)z^{-k} = F(z) \qquad (2.7)$$

Expression (2.7) is exactly the definition of one sided Z transform given previously by (2.2).

4. Some Properties of Z transform

Z transform, as Laplace transform, possess many properties, however in this section, we will focus on the most and widely used ones in the field of designing and analyzing discrete time (sampled data) control systems.

4.1. Linearity

The linearity property of Z transform is stated as follows:

Consider $f_1(k)$ and $f_2(k)$ being two discrete time (sampled) signals, for which:

$$\begin{cases} F_1(z) = Z\{f_1(k)\} \\ F_2(z) = Z\{f_2(k)\} \end{cases}$$

If α, β are real numbers, then:

$$Z\{\alpha f_1(k) \pm \beta f_2(k)\} = \alpha Z\{f_1(k)\} \pm \beta Z\{f_2(k)\} = \alpha F_1(z) \pm \beta F_2(z) \qquad (2.8)$$

4.2. Time Translation

If $f(k)$ is discrete time signal and $F(z)$ its Z transform, let $m \in \mathbb{N}$ and $T_s > 0$, then two types of time translation on $f(k)$ are distinguishable:

4.2.1. Time delay

$$Z\{f(k-mT_s)\} = z^{-m}Z\{f(k)\} = z^{-m}.F(z) \tag{2.9}$$

4.2.2. Time Advance

$$Z\{f(k+mT_s)\} = z^{m}\left[Z\{f(k)\} - \sum_{k=0}^{m-1} f(k).z^{-k}\right]$$

$$Z\{f(k+mT_s)\} = z^{m}\left[F(z) - \sum_{k=0}^{m-1} f(k).z^{-k}\right] \tag{2.10}$$

4.3. Time Multiplication

If $f(k)$ is discrete time signal and $F(z)$ its Z transform, the Z transform of the product time and the signal $f(k)$ is obtained as follows:

$$Z\{kf(k)\} = -zT_s\frac{d}{dz}[Z\{f(k)\}] = -zT_s\frac{dF(z)}{dz} \tag{2.11}$$

4.4. Discrete Convolution Theorem

If $f(k)$ is discrete time signal and $F(z)$ its Z transform, the Z transform of the time convolution of the two signals $f_1(k)$ and $f_2(k)$ is calculated as [8]:

$$Z\{f_1(k) * f_1(k)\} = F_1(z).F_1(z) \tag{2.12}$$

4.5. Initial value theorem

Let $f(k) = f(kT) = f^*(t)$ be a discrete time (sampled) signal and $F(z)$ its Z transform. If we are in the frequency domain and we do not know the time domain expression of the discrete signal, the calculation of its initial value is performed using the initial value theorem which is stated as follows:

$$f(0) = \lim_{k\to 0} f(k) = \lim_{z\to\infty} F(z) \tag{2.13}$$

4.6. Final Value Theorem

Let $f(k) = f(kT) = f^*(t)$ be a discrete time (sampled) signal and $F(z)$ its Z transform. If we are in the frequency domain and we do not know the time domain expression of the discrete signal, the calculation of its final value is performed using the final value theorem which is stated as follows:

$$f(\infty) = \lim_{k \to \infty} f(kT) = \lim_{z \to 1} (1 - z^{-1}) F(z) \qquad (2.14)$$

5. Z transform of the most familiar sampled signals

In the following table (**Table 2.1**), we summarize the Z transform of the most known and used discrete time signals regarding the design and analysis of sampled data control systems.

Table 2.1 Z transform of the most familiar discrete time signals

Continuous time signal: $f(t)$	**Discrete time signal: $f(kT_s)$**	**Z transform $F(z)$**
$\delta(t)$	$\delta(kT_s) = \delta(k)$	1
$u(t)$	$u(kT_s) = u(k)$	$\frac{z}{z-1}$
t	kT_s	$\frac{zT_s}{(z-1)^2}$
t^2	$(kT_s)^2$	$\frac{z(z+1){T_s}^2}{(z-1)^3}$
t^3	$(kT_s)^3$	$\frac{z(z^2+4z+1){T_s}^2}{(z-1)^4}$
e^{bt}	$e^{bkT_s} = a^k$	$\frac{z}{z-a} = \frac{z}{z-e^{bT_s}}$
a^t	a^{kT_s}	$\frac{z}{z-a^{T_s}}$
$1 - e^{\alpha t}$	$1 - a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
$e^{\alpha t} - e^{\beta t}$	$a^k - b^k$	$\frac{(a-b)z}{(z-a)(z-b)}$
$te^{-\alpha t}$	$kT_s a^k$	$\frac{azT_s}{(z-a)^2}$

$\sin(\omega_n t)$	$\sin(\omega_n kT_s)$	$\frac{\sin(\omega_n T_s)z}{z^2 - 2\cos(\omega_n T_s)\, z + 1}$
$\cos(\omega_n t)$	$\cos(\omega_n kT_s)$	$\frac{z[z - \cos(\omega_n T_s)]}{z^2 - 2\cos(\omega_n T_s)\, z + 1}$

6. Calculation Methods of Z transform

We distinguish two approaches to be used in calculating the Z transform of given signal. These are time domain and frequency domain approaches.

6.1. Time domain Approach

This approach considers the calculation of Z transform of a sampled signal directly using the definition of Z transform. That is, if $f(k) = f^*(t)$ is the sampled signal of the continuous time signal $f(t)$, its Z transform is calculated as:

$$F(z) = Z\{f(kT_s)\} = Z\{f^*(t)\} = L\{f^*(t)\}|_{z=e^{sT_s}}$$

$$F(z) = F^*(s)|_{z=e^{sT_s}} = \sum_{k=0}^{\infty} f(kT_s)z^{-k} \qquad (2.15)$$

This approach can be illustrated using the following diagram of Fig.2.2:

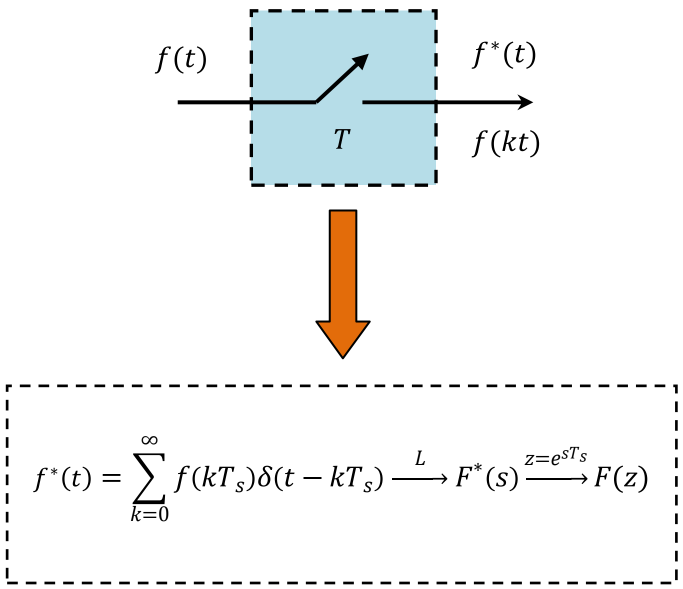


Fig.2.2 Diagram illustrating calculation of F(z) using time domain approach

Example:

Consider the sampled unit step signal $u(kT_s)$, we need to calculate its Z transfer function using the explained time domain approach.

Solution:

The use of time domain approach assumes the knowledge and availability of the sampled signal expression in the time domain.

The sampled unit step signal is defined as:

$$u(kT_s) = \begin{cases} 1, & \forall \ k \geq 0 \\ 0, & \forall \ k < 0 \end{cases}$$

Using (2.7), the Z transform of $u(k)$ is:

$$U(z) = Z\{u(kT_s)\} = U^*(s)|_{z=e^{sT_s}} = \sum_{k=0}^{\infty} u(kT_s).z^{-k} = \sum_{k=0}^{\infty} 1.z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

The latter expression is convergent geometric sequence of base: $r = z^{-1}$, it results:

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \qquad |z^{-1}| < 1$$

6.2. Frequency domain Approach

This approach assume that we only know the Laplace transform $F(s)$ of the continuous time signal $f(t)$, then we can calculate the Z transform $F(z)$ of the discrete time signal $f^*(t) = f(kT_s) = f(k)$.

In order to do so, several methods can be found; the most widely used are:

- ☞ Residues Method.
- ☞ Partial Fraction expansion Method.
- ☞ Polynomial Division Method.

We will explain the use of all them due to their importance and extensively use in both signal processing and digital control system design and analysis.

6.2.1. Residues Method

The Residues method of calculating Z transform of any sampled signal described by the Laplace transform of the corresponding continuous time signal is used as follows:

Given $F(s) = \mathcal{L}\{f(t)\}$, then, Z transform of $f^*(t)$ is calculated by:

$$F(z) = \sum_{p_i} R_i = \sum_{p_i} \left[Residues\ of\ \frac{F(s)}{1 - e^{sT_s} z^{-1}} \right] \Bigg|_{s=p_i} \qquad (2.16)$$

Where:

p_i: are the poles of $F(s)$.

R_i: are the residues corresponding to the poles p_i.

According to expression (2.16), the use of Residues method depends on two observed cases:

❖ Case 1: all the poles p_i are simple and real

If $N(s)$ and $D(s)$ are respectively the Numerator and Denominator of $F(s)$, that is we can write:

$$F(s) = \frac{N(s)}{D(s)} \qquad (2.17)$$

When all the poles p_i of $F(s)$ are simple and real, the Residue corresponding to the i^{th} pole is calculated as:

$$R_i = \left[Residue\ of\ \frac{F(s)}{1 - e^{sT_s} z^{-1}} \right] \Bigg|_{s=p_i} = \frac{N(p_i)}{D'(p_i)} \cdot \frac{1}{1 - e^{sT_s} z^{-1}} \Bigg|_{s=p_i} \qquad (2.18)$$

Where:

$$D'(p_i) = \frac{d[D(s)]}{ds} \Bigg|_{s=p_i} \qquad (2.19)$$

Illustration:

We give the following example to illustrate the application of Residues method in this first case of simple and real poles of the known Laplace transform function.

Let: $F(s) = \frac{1}{s}$, we want to calculate $F(z)$.

Answer:

We can write: $F(s) = \frac{N(s)}{D(s)} = \frac{1}{s}$, this means that: $\begin{cases} N(s) = 1 \\ D(s) = s \end{cases}$

The poles of $F(s)$ are the roots of $D(s)$; that is:

$$D(s) = 0 \Longrightarrow s = 0 \Longrightarrow s = p_1 = 0.$$

It results that $F(s)$ has only one simple and real pole.

Therefore:

$$D'(s = p_1) = \left.\frac{d[D(s)]}{ds}\right|_{s=p_1} = 1$$

Using (2.18), we obtain:

$$R_1 = \frac{N(p_i)}{D'(p_i)} \cdot \frac{1}{1 - e^{sT_s} z^{-1}} = \frac{1}{1} \cdot \left.\frac{1}{1 - e^{sT_s} z^{-1}}\right|_{s=0} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Since $F(s)$ has only one pole, it exists only one residue, consequently, the corresponding Z transform is:

$$F(z) = \sum_{p_i} R_i = R_1 = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

❖ Case 2: Repeated poles

When the Laplace transform $F(s)$ has a repeated pole, say p_i, of multiplicity factor « m », the Residue corresponding to this pole is calculated using the following general formula:

$$R_i = \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[(s - p_1)^r \frac{F(s)}{1 - e^{sT_s} z^{-1}} \right] \Bigg|_{s=p_1} \qquad (2.20)$$

Illustration:

Calculate the Z transform corresponding to the Laplace function given by:

$$F(s) = \frac{1}{s^2(s + 1)}$$

Answer:

We have: $F(s) = \frac{1}{s^2(s+1)} = \frac{N(s)}{D(s)}$, which implies that : $\begin{cases} N(s) = 1 \\ D(s) = s^2(s+1) \end{cases}$

We start by determining the poles of $F(s)$; this corresponds to solve the equation:

$$D(s) = 0 \Longrightarrow s^2(s+1) = 0 \Longrightarrow \begin{cases} s^2 = 0 \\ s+1 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} s = p_1 = 0 \ , (repeated\ pole\ of: m = 2) \\ s = p_2 = -1, \qquad simple\ and\ real\ pole \end{cases}$$

We calculate the residues corresponding to the repeated pole. Using (2.18) we get:

$$R_1 = \frac{d}{ds}\left[s^2 . \frac{1}{s^2(s+1)} \cdot \frac{1}{(1 - e^{sT_s}z^{-1})}\right]\Bigg|_{s=0}$$

$$= \left[\frac{-(1 - e^{sT_s}z^{-1}) + (s+1)T_s e^{sT_s}z^{-1}}{(s+1)^2(1 - e^{sT_s}z^{-1})^2}\right]\Bigg|_{s=0}$$

$$R_1 = \frac{-(1 - z^{-1}) + T_s z^{-1}}{(1 - z^{-1})^2} = -\frac{z}{z-1} + \frac{T_s z}{(z-1)^2}, \quad \forall\, T_s > 0$$

Using (2.18), the residue corresponding to the simple and real pole is:

$$R_2 = \frac{1}{D'(s)} \frac{1}{(1 - e^{sT_s}z^{-1})}\Bigg|_{s=-1} = \frac{1}{2s(s+1) + s^2} \frac{1}{(1 - e^{sT_s}z^{-1})}\Bigg|_{s=-1}$$

$$R_2 = \frac{1}{(1 - e^{-T_s}z^{-1})} = \frac{z}{z - e^{-T_s}}, \quad \forall\ T_s > 0$$

Regarding the two types of poles, the following end result can be obtained:

$$Z\left\{\frac{1}{s^2(s+1)}\right\} = R_1 + R_2 = -\frac{z}{z-1} + \frac{T_s z}{(z-1)^2} + \frac{z}{z - e^{-T_s}}, \quad \forall\, T_s > 0$$

6.2.2. Partial Fraction Expansion Method

This method is the most familiar and widely used particularly in the field of control system design and analysis. It consists of decomposing the Laplace transform function into simple fractions of known and easy determined Z transform. After we use the Z transform properties, it is possible to calculate Z transform of the originally given Laplace transform.

Assuming that Laplace transform function possesses both types of simple real poles and repeated poles, the following general formula is used to expand it into simple partial fractions.

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)^m . \prod_{j=m+1}^{n}(s-p_j)}$$

$$F(s) = \underbrace{\sum_{i=1}^{m} \frac{C_{1i}}{(s-p_1)^{m+1-i}}}_{\text{Simple Fractions Corresponding to Repeated poles}} + \underbrace{\sum_{j=m+1}^{n} \frac{C_j}{(s-p_j)}}_{\text{Simple Fractions Corresponding to Simple Real poles}} \qquad (2.21)$$

Where:

$F(s)$: represents Laplace transform fractional function of order "n".

p_1: is the repeated pole of $F(s)$ of multiplicity factor "m".

p_j: is the j^{th} simple and real pole of $F(s)$.

C_{1i} and C_j are unknown coefficients to be determined.

C_{1i} are the coefficients corresponding to the repeated poles; these are calculated using the following formula:

$$C_{1i} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} (s-p_1)^m F(s)|_{s=p_1} \ , \quad for: i = 1,2,3, \dots, m \qquad (2.22)$$

C_j: are the coefficients corresponding to the simple real poles, these are calculated using the following formula:

$$C_j = (s-p_j)F(s)\big|_{s=p_j} \qquad (2.22)$$

Example:

Let's take the Laplace transform function of the previous illustration and use partial fraction expansion to calculate its Z transform.

Answer:

By observing the s-function $F(s)$, it has the pole $p_1 = 0$ which repeated twice and one single pole $p_3 = -1$, which is simple and real. Using partial fraction expansion defined in general by (2.19), we obtain:

$$F(s) = \frac{1}{s^2(s+1)} = \frac{C_{11}}{s} + \frac{C_{12}}{s^2} + \frac{C_3}{(s+1)}$$

Using (2.20) and (2.21), the coefficients C_{11}, C_{12} and C_3 are respectively calculated as:

$$C_{11} = \frac{d}{ds}[s^2\mathrm{F(s)}]|_{s=0} = \frac{d}{ds}\left[\frac{1}{s+1}\right]\Big|_{s=0} = -1$$

$$C_{12} = [s^2\mathrm{F(s)}]|_{s=0} = \left[\frac{1}{s+1}\right]\Big|_{s=0} = 1$$

$$C_3 = [(s+1)\mathrm{F(s)}]|_{s=-1} = \left[\frac{1}{s^2}\right]\Big|_{s=-1} = 1$$

Hence :

$$F(s) = \frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{(s+1)}$$

By applying Z transform, we have:

$$F(z) = Z[F(s)] = Z\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{(s+1)}\right] = -TZ\left[\frac{1}{p}\right] + TZ\left[\frac{1}{p^2}\right] + TZ\left[\frac{1}{(p+1)}\right]$$

Using linearity property of Z transform, it turns out that:

$$F(z) = -Z\left[\frac{1}{s}\right] + Z\left[\frac{1}{s^2}\right] + Z\left[\frac{1}{(s+1)}\right]$$

To finish, we still only to use the correspondence table between Laplace transform and Z transform of the basic signals to find:

$$F(z) = -\frac{z}{z-1} + \frac{T_s z}{(z-1)^2} + \frac{z}{z-e^{-T_s}}, \quad \forall\, T_s > 0$$

Obviously, identical results are obtained using either Residues or Partial fraction expansion methods.

7. Inverse Z Transform

7.1. Definition

While Z transform represents a mathematical tool applied on discrete time (sampled) signal which allows us to transform the work domain from time domain into frequency domain, inverse Z transform is the reverse operation; in other words, it is a transformation that allows us to obtain the discrete time domain representation of a sampled signal from the knowledge of its frequency domain representation.
The inverse Z transform mechanism, denoted by the symbol $\mathcal{Z}^{-1}$, is described as:

$$if: \;\; F(z) = \mathcal{Z}\{f^*(t)\} = \mathcal{Z}\{f(kT)\}$$

$$then:$$

$$f^*(t) = f(kT) = \mathcal{Z}^{-1}[F(z)] \qquad (2.23)$$

7.2. Calculation of Inverse Z transform

The same methods which are explained earlier and used to calculate Z transform of discrete time signals are also employed in calculating the inverse Z transform with only a little bit difference in their formulation and application procedure. These are as follows:

7.2.1. Residues Method

If it is known the Z transform $F(z) = Z\{f^*(t)\}$, the sampled (discrete time) signal $f^*(t) = f(kT)$ is obtained back using residues method according to the following formula:

$$f^*(t) = f(kT) = \sum_{p_i} R_i = \sum_{p_i} [Residue\ de\ z^{k-1}.F(z)]|_{z=p_i} \qquad (2.24)$$

Where:

p_i: are the poles of the Z transform function $F(z) = \frac{N(z)}{D(z)}$

R_i: is the residue corresponding to the pole p_i of multiplicity factor "m" , which is calculated using the following general formula :

$$R_i = \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} [(z-p_i)^m \, z^{k-1}.F(z)]|_{z=p_i} \qquad (2.24)$$

To illustrate the use of this method, we consider the following example.

Example:

Consider the Z transform defined by the function $F(z) = \frac{T_s.z}{(z-1)^2}$, $\forall\, T_s > 0$.

Calculate the inverse Z transform of $F(z)$?

Answer:

First of all, we need to determine the poles of $F(z)$. we have:

$$(z-1)^2 = 0 \Longrightarrow z = p_1 = 1\!: repeated\ pole\ of\ m = 2$$

By applying the Residues defined by (2.24), we get:

$$f^*(t) = f(kT) = \sum\nolimits_{p_i} R_i = R_1$$

With :

$$R_1 = [Residue\ de\ z^{k-1}.F(z)]|_{z=p_i} = \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} [(z-p_i)^m \, z^{k-1}.F(z)]|_{z=p_i}$$

For $m = 2$:

$$R_1 = \frac{1}{(2-1)!} \frac{d^{(2-1)}}{dz^{(2-1)}} [(z-1)^2 \, z^{k-1}.F(z)]|_{z=1}$$

$$R_1 = \frac{d}{dz}\left[(z-1)^2 \, z^{k-1}.\frac{T_s z}{(z-1)^2}\right]\Bigg|_{z=1} = \frac{d}{dz}[T_s z^k]|_{z=1} = kT_s[z^{k-1}]|_{z=1} = kT_s$$

Therefore :

$$f^*(t) = f(kT_s) = \mathcal{Z}^{-1}[F(z)] = R_1 = kT_s, \quad \forall\, T_s > 0$$

Which represents the sampled unit ramp signal.

7.2.2. Partial Fraction Expansion Method

This method, as it is previously explained, is based on decomposing the fractional z-function into simple elementary fractions where each partial fraction is a Z transform of one of the basic and familiar sampled signals. However, the use of this method in calculating the inverse Z transform of $F(z)$ is done according to the following procedure:

Step 1: we construct the function: $\frac{F(z)}{z}$

Step 2: use partial fraction expansion to expand $\frac{F(z)}{z}$ into simple elementary fractions.

Step 3: obtain again the original function $F(z) = \frac{F(z)}{z} \times z$

Step 4: apply the inverse Z transform definition and use Z transform table to obtain the discrete time signal $f(kT_s)$.

Illustration 2.3

Given the z- function described by: $F(z) = \frac{z}{4z^2-5z+1}$.

Use the partial fraction expansion method to calculate the sampled signal $f(kT_s)$.

Answer:

We shall follow the indicated procedure according to the shown steps.

Step 1: construction of the function $\frac{F(z)}{z}$

We have :

$$\frac{F(z)}{z} = \frac{z}{4z^2 - 5z + 1} \times \frac{1}{z} = \frac{1}{4z^2 - 5z + 1} = \frac{1}{(z-1)\left(z - \frac{1}{4}\right)}$$

Step 2 : expansion of $\frac{F(z)}{z}$ into simple fractions

$$\frac{F(z)}{z} = \frac{1}{(z-1)\left(z - \frac{1}{4}\right)} = \frac{C_1}{(z-1)} + \frac{C_2}{\left(z - \frac{1}{4}\right)}$$

Where :

$$C_1 = \left[(z-1)\frac{\mathrm{F(z)}}{\mathrm{z}}\right]\Big|_{z=1} = \left[(z-1)\frac{1}{(z-1)(z-\frac{1}{4})}\right]\Bigg|_{z=1} = \left[\frac{1}{(z-\frac{1}{4})}\right]\Bigg|_{z=1} = \frac{4}{3}$$

$$C_2 = \left[\left(z-\frac{1}{4}\right)\frac{\mathrm{F(z)}}{\mathrm{z}}\right]\Bigg|_{z=\frac{1}{4}} = \left[\left(z-\frac{1}{4}\right)\frac{1}{(z-1)(z-\frac{1}{4})}\right]\Bigg|_{z=\frac{1}{4}} = \left[\frac{1}{(z-1)}\right]\Bigg|_{z=\frac{1}{4}} = -\frac{4}{3}$$

Therefore:

$$\frac{F(z)}{z} = \frac{1}{(z-1)\left(z-\frac{1}{4}\right)} = \frac{\frac{4}{3}}{(z-1)} - \frac{\frac{4}{3}}{\left(z-\frac{1}{4}\right)}$$

Step 3: obtaining $F(z)$

$$F(z) = \frac{F(z)}{z} \times z = \frac{\frac{4}{3}z}{(z-1)} - \frac{\frac{4}{3}z}{\left(z-\frac{1}{4}\right)}$$

Then:

$$f(k) = f(kT) = \mathcal{Z}^{-1}\{F(z)\} = \mathcal{Z}^{-1}\left\{\frac{\frac{4}{3}z}{(z-1)} - \frac{\frac{4}{3}z}{\left(z-\frac{1}{4}\right)}\right\}$$

$$f(k) = \frac{4}{3}\mathcal{Z}^{-1}\left\{\frac{z}{(z-1)}\right\} - \frac{4}{3}\mathcal{Z}^{-1}\left\{\frac{z}{\left(z-\frac{1}{4}\right)}\right\}$$

Step 4: using Z transform table of standard signals

By referring to Z transform table of standard sampled signals we find:

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-1)}\right\} = u(kT_s)\,, \qquad \forall\, T_s > 0$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{\left(z-\frac{1}{4}\right)}\right\} = \left(\frac{1}{4}\right)^{kT_s}, \qquad \forall\, T_s > 0$$

It results:

$$f(k) = f(kT) = \frac{4}{3}u(kT_s) - \frac{4}{3}\left(\frac{1}{4}\right)^{kT_s}, \qquad \forall\, T_s > 0$$

7.2.3. Polynomial Division Method

Polynomial division, also called long division, is based on one sided definition of Z transform given by (2.2) obtained after performing polynomial division of the Numerator $N(z)$ over Denominator $D(z)$ of the function $F(z)$. The result of this division gives rise the time sequence defining the sampled signal; that is if:

$$F(z) = \mathcal{Z}\{f(kT_s)\} = \mathcal{Z}\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT_s) z^{-k}$$

$$\Rightarrow f(kT_s) = \mathcal{Z}^{-1}\{F(z)\} = \{f(0), f(1T_s), f(2T_s), f(3T_s), \dots \dots ..\}$$

The number of samples in the obtained time sequence is determined in such a way sufficient data points are reached.

Example:

Consider the function defined as:

$$F(z) = \frac{2z+3}{z^2 - 0.4z + 0.2}$$

We want to calculate the inverse Z transform using the polynomial division method.

Solution:

$$2z+3 \left| \underbrace{z^2 - 0.4z + 0.2}_{2z^{-1}+3.8z^{-2}+1.12z^{-3}+\cdots \dots} \right.$$

The result of division yields:

$$F(z) = 2z^{-1} + 3.8z^{-2} + 1.12z^{-3} + \cdots \dots$$

From which the inverse Z transform is:

$$f(kT_s) = \mathcal{Z}^{-1}\{F(z)\} = \{f(0), f(1T_s), f(2T_s), f(3T_s), \dots \dots ..\}$$

$$f(kT_s) = \{0, 2, 3.8, 1.12, \dots \dots ..\}$$