Chapter 1: General Structure of Discrete Time Control System

Digital systems such as computers operate on digital signals; accordingly, the need for handling digital signals is also increased. Due to high-speed processing capabilities of modern digital systems, wide range of applications make use of digital signals, which further accelerate the development of the use of digital signals. Hence, digital control systems have gradually become more prominent in today's industries.

In this chapter we will give the main definitions relating to sampled and discrete signals. We also explore the operation and procedure of obtaining discrete and sampled signals from the corresponding continuous time signals. We also give both mathematical and graphical representation of the basic and standard discrete signals which are widely employed in designing and analyzing a sampled data control systems. We finish the chapter by describing the general structure and block diagram of a typical discrete time feedback control system.

1. Basic Definitions

1.1. What is a signal?

A signal can be defined as a physical quantity that is generated by the evolution of an engineering process. It carries information and measurement about the behavior of that process. For example: the temperature, pressure, voltage, current, sound, etc. It is however important to know that any signal as a physical quantity evolves in time with a variable amplitude. Hence it can be represented as a function where the amplitude takes different values at different corresponding time instants.

1.2. Continuous time (Analog) signal

A continuous time signal is defined as a signal which is both continuous in time and amplitude. Sometimes, a continuous signal is also called analog signal and both names are used interchangeably. The continuous (analog) signal is originally found in nature and is usually measured and generated by the different sensors.

1.3. Discrete time (digital) signal

In any data acquisition system, the measured values of the physical quantities are taken at regular periods of time. Therefore, the time is never being considered as continuous but discrete. As such, a discrete time signal is defined as that obtained by converting a continuous signal at discrete instants of time. This is called discretization of continuous signal; hence the main property of a discrete signal is that it takes finite amplitude values at discrete instants of time.

1.4. Sampled signal

A sampled signal is defined as a continuous signal which is discretized at a regular time step and interval called sampling period. We will come back to explain in more details in the foregoing sections of this chapter.

1.5. Quantification

The quantification is defined as the process and operation of attributing to each amplitude value of the sampled signal a binary number; that is a pattern of bits 0s and 1s.

1.6. Digital signal

A digital signal can be defined as a sampled signal with quantified amplitude.

1.7. Causal signal

A causal signal is defined to be of zero values for negative instants of time. This signal is of paramount importance for the design and analysis of control systems in general. Therefore, regarding the scope of course, we will be interesting only with causal signals and causal systems.

2. Laplace transform: A Review

2.1. Definition

Laplace transform is a mathematical tool used to fundamentally solve linear differential equations. In the context of control system design and analysis, this tool is employed to develop and derive a continuous input-output modeling and representation of the system (process) to be controlled or the whole control system. The use of Laplace transform also makes ease a direct and qualitative analysis of the effect of different environmental variables and parameters of the system's behavior and performance.

Mathematically, the mapping of the continuous time function $f(t)$ of the variable 't' to the frequency domain function $F(s)$ of the complex variable 's' is called Laplace transform. It is defined as [1]:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t).e^{-st}\,dt \tag{1.1}$$

For $f(t)$ causal signal, that is: $f(t) = 0,\ for\ \ t < 0$, Laplace transform definition becomes:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t).e^{-st}\,dt \tag{1.2}$$

Where:

$\mathcal{L}\{.\}$: is the symbol given to Laplace transform operator.

Expressions (1.1) and (1.2) define respectively two-sided and one-sided Laplace transforms.

In both expressions, we read that Laplace transform of the continuous time signal $f(t)$ is $F(s)$.

Example:

Let $f(t) = e^{-at}$ be continuous and causal signal.

Calculate its Laplace transform.

Answer:

We apply the definition (1.2) on the signal $f(t)$, we get :

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t).e^{-st}\,dt = \int_{0}^{\infty} e^{-at}.e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt$$

$$F(s) = \frac{1}{-(s+a)}\left[e^{-(s+a)t}\right]_0^{\infty}$$

Therefore:

$$\mathcal{L}\{f(t) = e^{-at}\} = F(s) = \frac{1}{(s+a)}$$

2.2. Properties of Laplace Transform

Many important properties are characterizing Laplace transform. If we have:

$$F(s) = \mathcal{L}\{f(t)\} \tag{1.3}$$

We summarize the main of these properties in **Table 1.1** as follows:

Table 1.1 Main properties of Laplace transform

N°	Property	Time signal	Laplace Transform
1	Linearity	$\alpha f_1(t) \pm \beta f_2(t)$	$\alpha F_1(s) \pm \beta F_2(s)$
2	Time Delay	$f(t-m)$	$e^{-ms}.F(s)$
3	Time Advance	$f(t+m)$	$e^{ms}.F(s)$
4	Complex Translation	$e^{+at}.f(t)$	$F(s-m)$
		$e^{-at}.f(t)$	$F(s+m)$
5	Time Scaling	$f(t/a)$	$aF(as)$
6	Derivative	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^+)$
7	Derivative of Order n	$\dfrac{d^{(n)}f(t)}{dt^n}$	$s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} . \left.\dfrac{d^{(k)}f(t)}{dt^k}\right\vert_{t=0^+}$
8	Integral	$\int_0^t f(\tau)\, d\tau$	$\dfrac{F(s)}{s} + \dfrac{f^{-1}(0^+)}{s}$
9	Initial value theorem	$f(0^+) = \lim_{t\to 0^+} f(t)$	$\lim_{s\to +\infty} sF(s)$
10	Final value theorem	$f(\infty) = \lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$

2.3. Laplace transform of basic and familiar signals

In the following table (**Table 1.2**), we mention and summarize the Laplace transform of the most basic signals and widely employed in the design and analysis of continuous time control systems.

Table 1.2 Laplace transform of basic signals

N°	**Time domain function : $f(t)$**	**Laplace Transform: $F(s)$**
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$t.u(t)$	$\frac{1}{s^2}$
4	$t^n.u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}.u(t)$	$\frac{1}{s+a}$
6	$t^n.e^{-at}.u(t)$	$\frac{1}{(s+a)^{n+1}}$
7	$\sin(\omega t).u(t)$	$\frac{\omega}{s^2+\omega^2}$
8	$\cos(\omega t).u(t)$	$\frac{s}{s^2+\omega^2}$
9	$e^{-at}\sin(\omega t).u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
10	$e^{-at}\cos(\omega t).u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Where:

$\delta(t)$: is the unit impulse (Dirac) signal (function).

$u(t)$: is the unit step signal (function).

3. Sampling of Continuous (Analog) signals

3.1. Definition of Sampling

The sampling can be defined as the process and operation of converting a continuous time (analog) signal $f(t)$ into discrete time signal consisting of a series of impulses of amplitudes determined as the values of the continuous time signal $f(t)$ at the sampling instants [2]. Consequently, the sampling operation produces a sequence of samples denoted by $\{f(kT_s)\}$ from the given analog signal $f(t)$. We express this sequence of samples as:

$$\{f(kT_s)\} = \{f(0), f(1T_s), f(2T_s), \dots., f(kT_s)\} \tag{1.4}$$

We denote the sampled signal as;

$$\begin{aligned} f^*(t) &= f(kT_s) = \{f(kT_s)\} \\ &= \{f(0), f(1T_s), f(2T_s), \dots, f(kT_s)\} = \{f_0, f_1, f_2, \dots, f_k\} \end{aligned} \tag{1.5}$$

Where:

k: is an integer ; $k \in \mathbb{N}$.

T_s: is the sampling period ($T_s > 0$).

kT_s: are defined to be the sampling time instants.

$f(kT_s) = f_k = f^*(t)$: is the notation used to refer the sampled signal obtained to the amplitudes of the continuous time signal $f(t)$ at the corresponding sampling time instants kT_s.

3.2. Principle of Operation of Sampling

To understand the principle of operation of sampling, we need to define the following two important functions:

3.2.1. Dirac (Impulse) Function: $\delta(t)$

The Dirac function $\delta(t)$, also known as Kronecker function, is a non-practical signal. Mathematically, it is defined to be a rectangular signal for which the time width tends to zero and the amplitude (length) tends to infinity with an area equals one. This is expressed as [3]:

$$\delta(t) = \begin{cases} \infty, & if \;\; t = 0 \\ 0, & \forall \;\; t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = 1 \end{cases} \tag{1.6}$$

As a shifted version of $\delta(t)$ at the time instant t_0, $\delta(t-t_0)$ is defined by the following:

$$\delta(t-t_0) = \begin{cases} \infty, & if \;\; t = t_0 \\ 0, & \forall \;\; t \neq t_0 \\ \int_{-\infty}^{\infty} \delta(t)dt = 1 \end{cases} \tag{1.7}$$

3.2.2. Train of Impulses: $\delta_T(t)$

The train of impulses, denoted as $\delta_T(t)$**,** is known to be the sampling function or the sampler in the field of sampling. Mathematically it is defined as:

$$P(t) = \delta_{T_s}(t) = \sum_{k=0}^{\infty} \delta(t-kT_s) \tag{1.8}$$

Graphically, the sampling function is represented, using Matlab, as it is shown in the figure below:

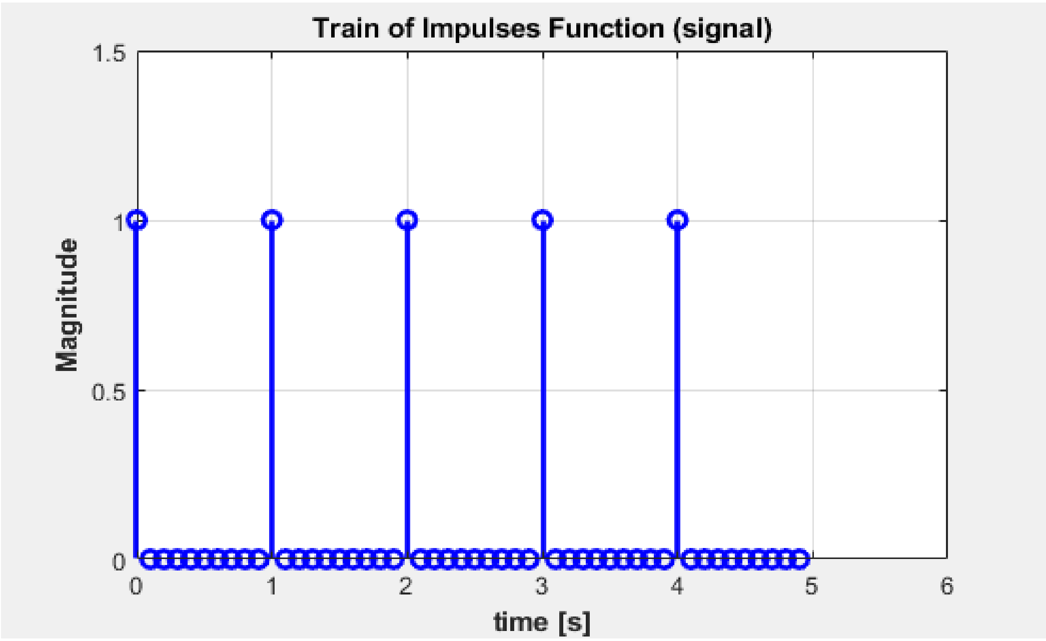

Fig.1.1 Graphical representation of train of impulses signal

Based on the above definitions of the Dirac and the train of impulses function as basic signals, the sampling of an analog signal $f(t)$ can be mathematically described as the product and multiplication of the continuous signal to be sampled by the train of impulses (sampler), $P(t)$.

For $t > 0$, we can write:

$$f^*(t) = f(kT) = f(k) = f(t) \times P(t) = f(t) \times \delta_T(t)$$

$$f^*(t) = f(t) \times \sum_{k=0}^{\infty} \delta(t - kT) = \sum_{k=0}^{\infty} f(kT) \times \delta(t - kT) \qquad (1.9)$$

We can illustrate the sampling operation as it is shown in the following figure:

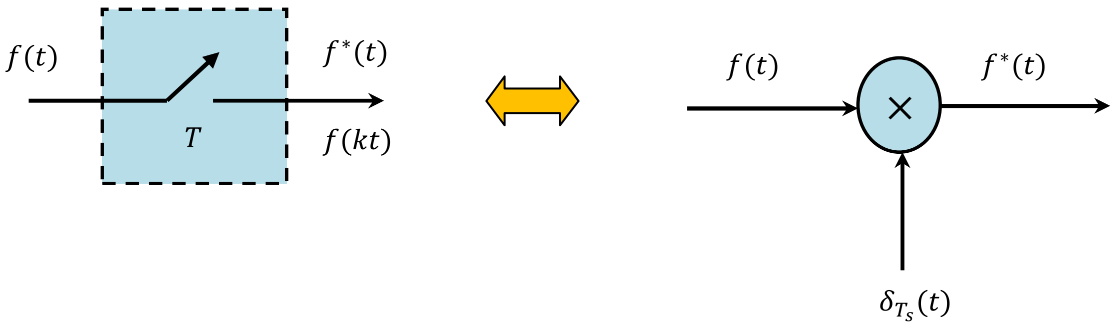

Fig.1.2 Schematic illustration of sampling

3.2.3. Modeling and Representation of a Sampler

For the sake of simplifying the study and analysis of sampled data control systems, the sampling mechanism depicted and illustrated in **Fig.1.2** above is conveniently modeled and represented by an ideal switch that closes at each sampling instant for an infinitesimal time duration ($t \to 0$) and keep open thereafter for a time duration corresponding to the value of the sampling period T_s. This model given to the sampler is shown in the following figure (**Fig.1.3**).

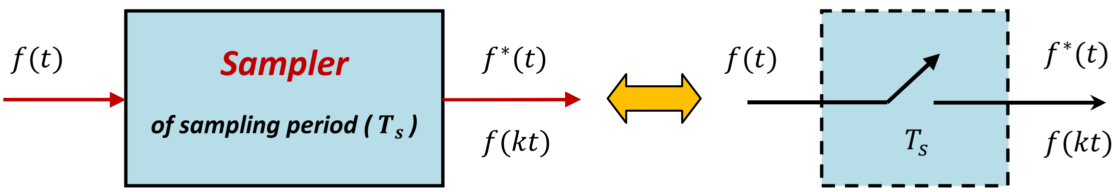

Fig. 1.3 Modeling of a sampler using an ideal switch

3.2.4. Typical standard sampled signals

The outcome of the sampling operation described above is the sampled signal or a discrete time signal. To end up this section, we give some examples of the most familiar and widely used discrete time signals as long as the sampled data control systems are concerned. It is important to notice that only causal signals are being described.

3.2.4.1. Unit impulse discrete time signal: $\delta(kT_s)$

The sampled or discrete time of the impulse or Dirac signal is defined as:

$$\delta(k) = \begin{cases} 1, & if \;\; k = 0 \\ 0, & \forall \;\; k \neq 0 \end{cases} \tag{1.10}$$

However, the discrete time unit impulse signal shifted in time at k_0 is defined as:

$$\delta(k - k_0) = \begin{cases} 1, & if \;\; k = k_0 \\ 0, & \forall \;\; k \neq k_0 \end{cases} \tag{1.11}$$

Graphically, both discrete time unit impulse signal and its time shift version are represented according to the figure below:

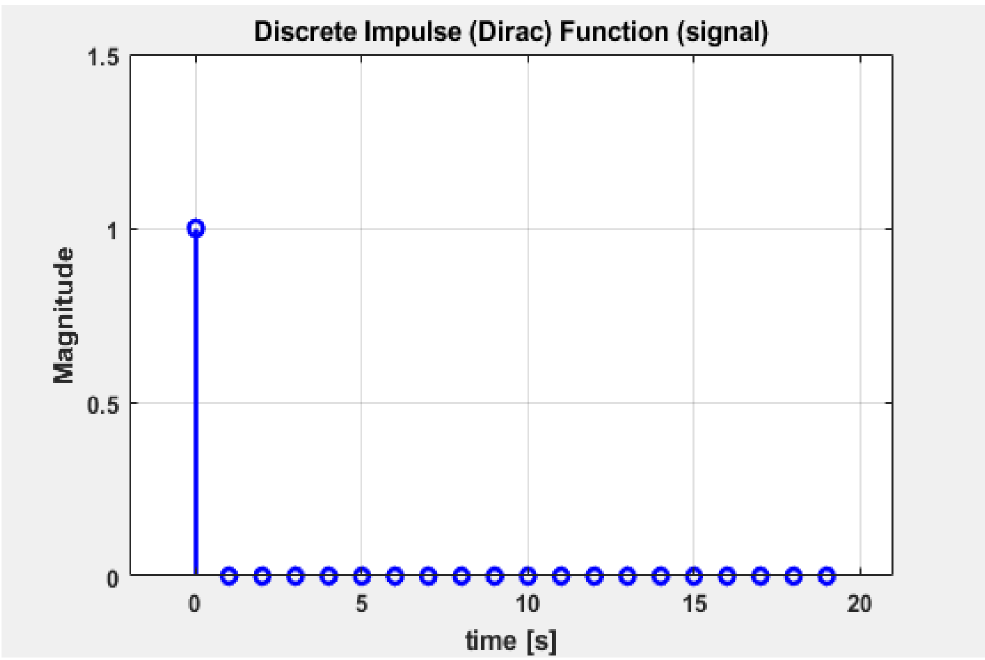

Fig.1.4 Graphical representation of discrete impulse signal

3.2.4.2. Unit step discrete time signal: $u(k)$

When sampled, the unit step signal is defined as:

$$u(k) = \begin{cases} 1, & \forall\ k \geq 0 \\ 0, & \forall\ k < 0 \end{cases} \tag{1.12}$$

The sequence definition of the discrete time (sampled) unit step signal is given as:

$$u(k) = u(kT_s) = u^*(t) = \left\{ \underset{\uparrow k=0}{\underbrace{1}} ,1,1,1,1, \ldots 1 \right\} \tag{1.13}$$

Similarly, we can define a time shift at k_0 of the above unit step signal as:

$$u(k - k_0) = \begin{cases} 1, & \forall\ k \geq k_0 \\ 0, & \forall\ k < k_0 \end{cases} \tag{1.14}$$

In the following figure, we show the graphical representation of the sampled unit step signal as well as its time shifted version, where the sampling period is taken arbitrary as: $T_s = 1\ sec.$

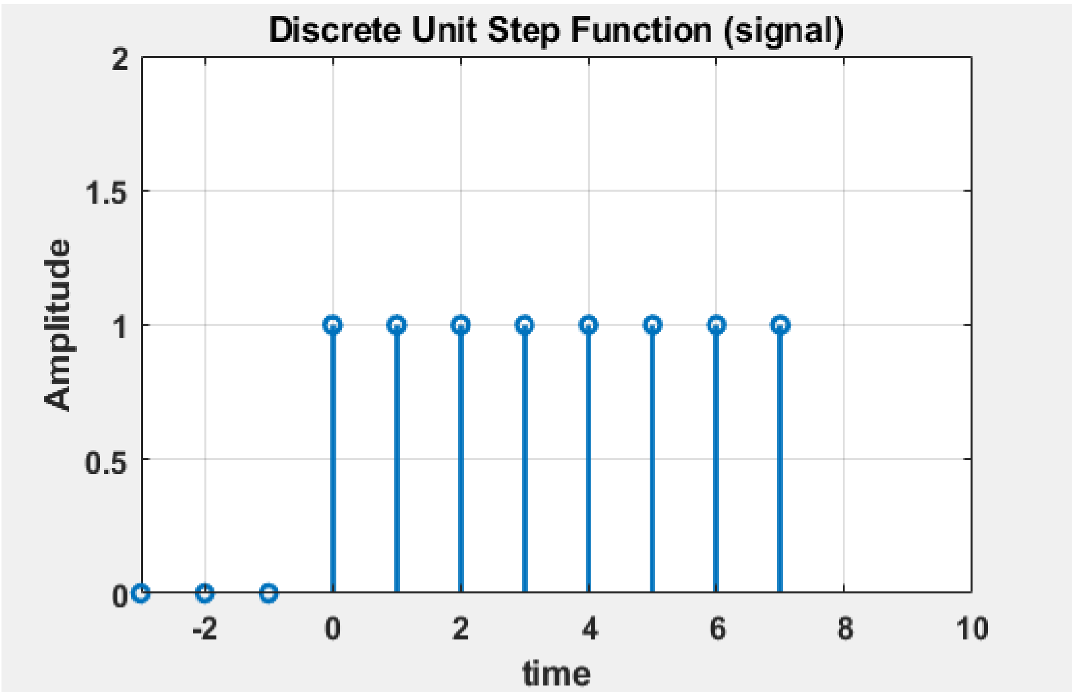

Fig.1.5 Graphical representation of discrete unit step signal

3.2.4.3. Unit Ramp discrete time signal: $r(k)$

The unit ramp sampled signal is defined according to the following description:

$$r(k) = \begin{cases} k, & \forall\ k \geq 0 \\ 0, & \forall\ k < 0 \end{cases} \quad or: \quad r(k) = \begin{cases} kT_s, & \forall\ k \geq 0 \\ 0, & \forall\ k < 0 \end{cases} \tag{1.15}$$

As a sequence, it is defined as:

$$r(k) = r(kT_s) = r^*(t) = \sum_{k=0}^{\infty} r(kT_s)\delta(t - kT_s)$$

$$r(k) = \sum_{k=0}^{\infty} (kT_s)\delta(t - kT_s) = \left\{ \underbrace{0}_{\uparrow k=0}, 1,2,3,4, \dots k \right\} \tag{1.16}$$

When it is time shifted at k_0, it is defined as :

$$r(k - k_0) = \begin{cases} k, & \forall\ k \geq k_0 \\ 0, & \forall\ k < k_0 \end{cases} \tag{1.17}$$

Using Matlab, the graphical representation of a typical discrete ramp signal is shown in **Fig.1.6** below, where the sampling period is taken to be: $T_s = 1\ sec$.

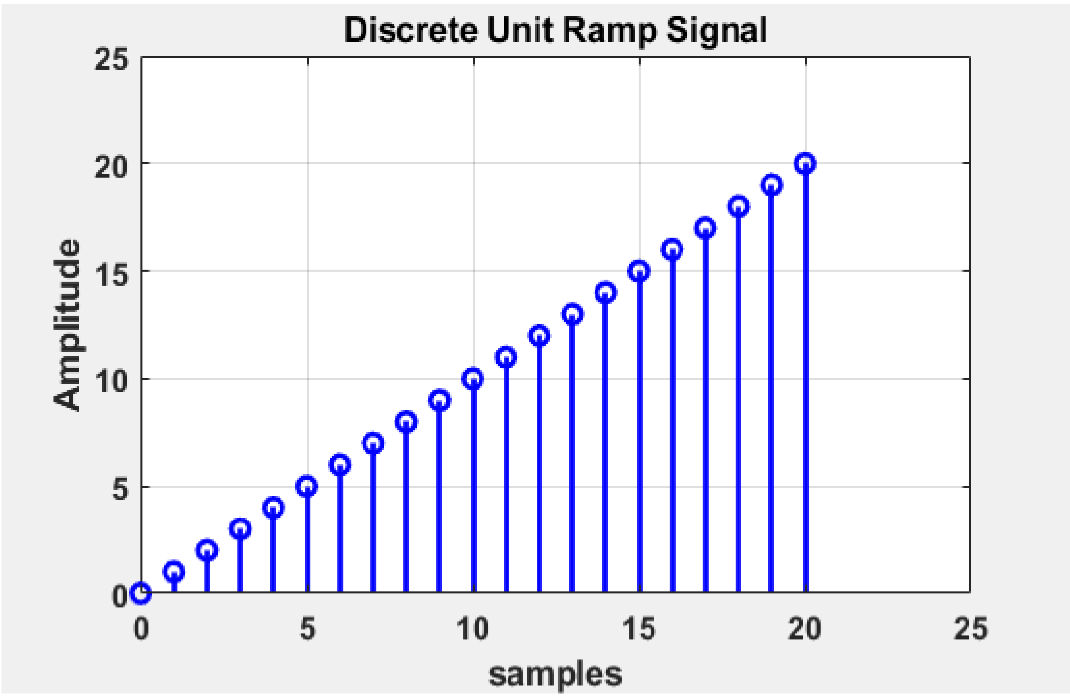

Fig.1.6 Graphical representation of discrete ramp signal

3.2.4.4. Sinusoidal Discrete time signal

If we consider $f(t)$ being unity amplitude continuous time and causal sinusoidal signal defined as:

$$f(t) = sin(t), \forall t \geq 0 \tag{1.18}$$

When sampled at a regular sampling period T_s, the obtained discrete time signal is defined as:

$$f^*(t) = f(kT_s) = f(k) = \sum_{k=0}^{\infty} f(kT_s)\delta(t - kT_s)$$

$$f^*(t) = \sum_{k=0}^{\infty} \sin(\omega kT_s)\delta(t - kT_s) = \begin{cases} \sin(\omega kT_s), & \forall\ k \geq 0 \\ 0, & \forall\ k < 0 \end{cases} \tag{1.19}$$

Using MATLAB, the sampled sinusoidal signal can be depicted in **Fig.1.7**. A comparison is done with the corresponding continuous time sinusoidal signal. We also notice that the sampling period is taken in this case to be: $T_s = 1\ sec$.

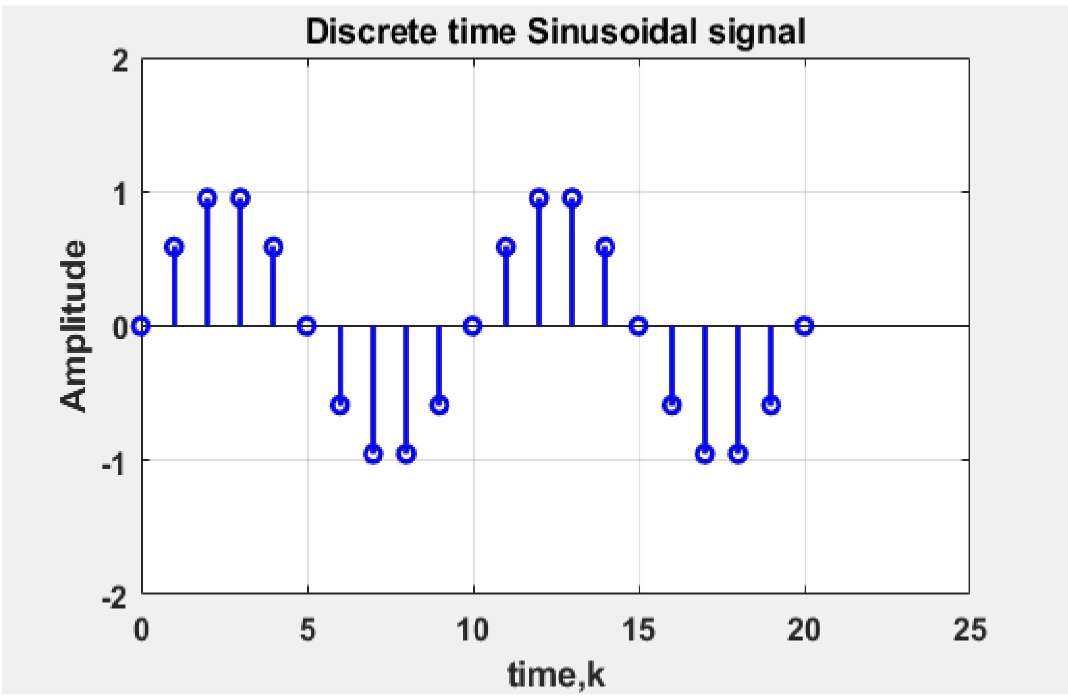

Fig.1.7 Graphical representation of discrete time sinusoidal signal

4. Selecting the Appropriate Sampling Period

It is known in digital signal processing domain that applying sampling operation to convert continuous time (analog) signal into discrete time (digital) signal is accompanied by an inherent error which is attributed to the choice of the sampling period T_s. Unfortunately, this error causes an important loss of information contained in the signal if the value of the sampling period is not conveniently chosen. This problem is also encountered when the design and analysis of digital control system is concerned. In order to solve this problem and achieving a good sampling, Shannon theorem is applied to choose the desired and convenient the sampling period. Using this theorem, it is ensured that the sampling operation is performed with minimum loss of information.

4.1. Shannon theorem

Shannon theorem states that for a continuous time signal to be built and regenerated from a given sequence of samples with a sampling period T_s, the sampling angular frequency defined as $\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$ must be at least two times greater than the angular frequency of the of the continuous time signal. This statement can be interpreted as:

If we assume $f(t)$ to be a continuous time signal of finite energy, hence having a calculated Fourier transform denoted by $F(\omega)$ and illustrated as shown in **Fig.1.8** below**:**

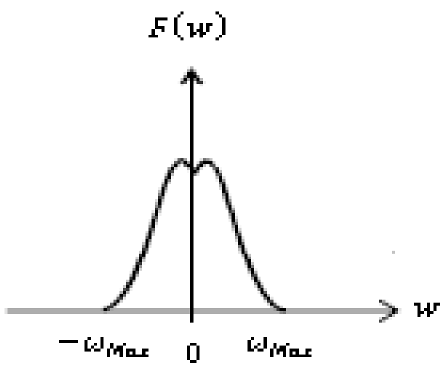

Fig.1.8 Frequency spectral of a given continuous time signal $f(t)$

Where**:**

ω: is the angular frequency of the continuous time signal.

ω_{Max}: is the greatest angular frequency contained in the frequency spectral $F(\omega)$.

Then the Fourier transform $F^*(\omega)$ of the sampled signal $f^*(t)$ can be defined as:

$$F^*(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s) \tag{1.20}$$

We notice that $F^*(\omega)$ is an infinite frequency spectrum obtained from the periodic spectrum of $F(\omega)$ around the sampling angular frequency ω_s.

By applying Shannon theorem, a perfect sampling operation is obtained if the sampling period is chosen such that the following relationship is satisfied:

$$\omega_s \geq 2\omega_{Max} \tag{1.21}$$

The illustration of the frequency spectrum of the sampled signal $F^*(\omega)$ is shown in the following **Fig. 1.9**.

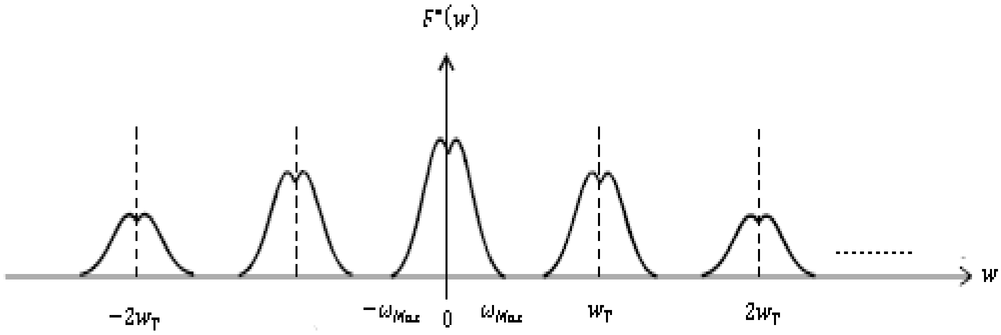

Fig. 1.9 Case of frequency spectrum $F^*(\omega)$ without aliasing phenomenon

The effect of sampling period on the performance and operation of the digital control system can be seen from the following practical issues [2]:

- When performing sampling, it is required to apply and respect the Shannon theorem.
- If the sampling period T_s is too small, the robustness and disturbance rejection performance of the feedback control system is highly deteriorated.
- If on the other hand the sampling period T_s is sufficiently high, high memory storage is meaninglessly needed.

In the following table (**Table 1.3**), it is given the appropriate values of the sampling period regarding some industrial processes subjected to sampling operation [8].

Table 1.3 Typical sampling period values for some processes

Signal	*Recommended sampling period T_s*
Current in Electric Drives	$50 < T_s < 100\ (\mu s)$
Position in Robotics	$0.2 < T_s < 1\ (ms)$
Position in Machine Tools	$0.5 < T_s < 10\ (ms)$
Rate Signal	$1 < T_s < 3\ (s)$
Level Signal	$5 < T_s < 10\ (s)$
Pressure	$1 < T_s < 5\ (s)$
Temperature	$10 < T_s < 45\ (s)$

5. Typical block diagram of discrete time feedback control system

Throughout the previous sections and subsections of this chapter, the main and basic notions and concepts which are used in the context of discrete time (sampled data) control systems are defined and described. However, the crucial idea in all of these concepts is the sampling operation which allows the conversion of analog (continuous time) signals into discrete time signals. Due to the fact that the majority the controlled processes are of analog nature, the use and implementation of these signal converters are principal and mandatory to be able of performing the desired control tasks.

In order to accommodate the functional and operational requirements in terms of signal type processing, any discrete time (digital) control system should incorporate these signal conversion devices. These are called respectively Analog-to-Digital and Digital-to-Analog Converters, which are respectively abbreviated as ADC and DAC.

Consequently, the general structure of the block diagram representing a typical discrete time feedback control system is as illustrated in **Fig.1.10**.

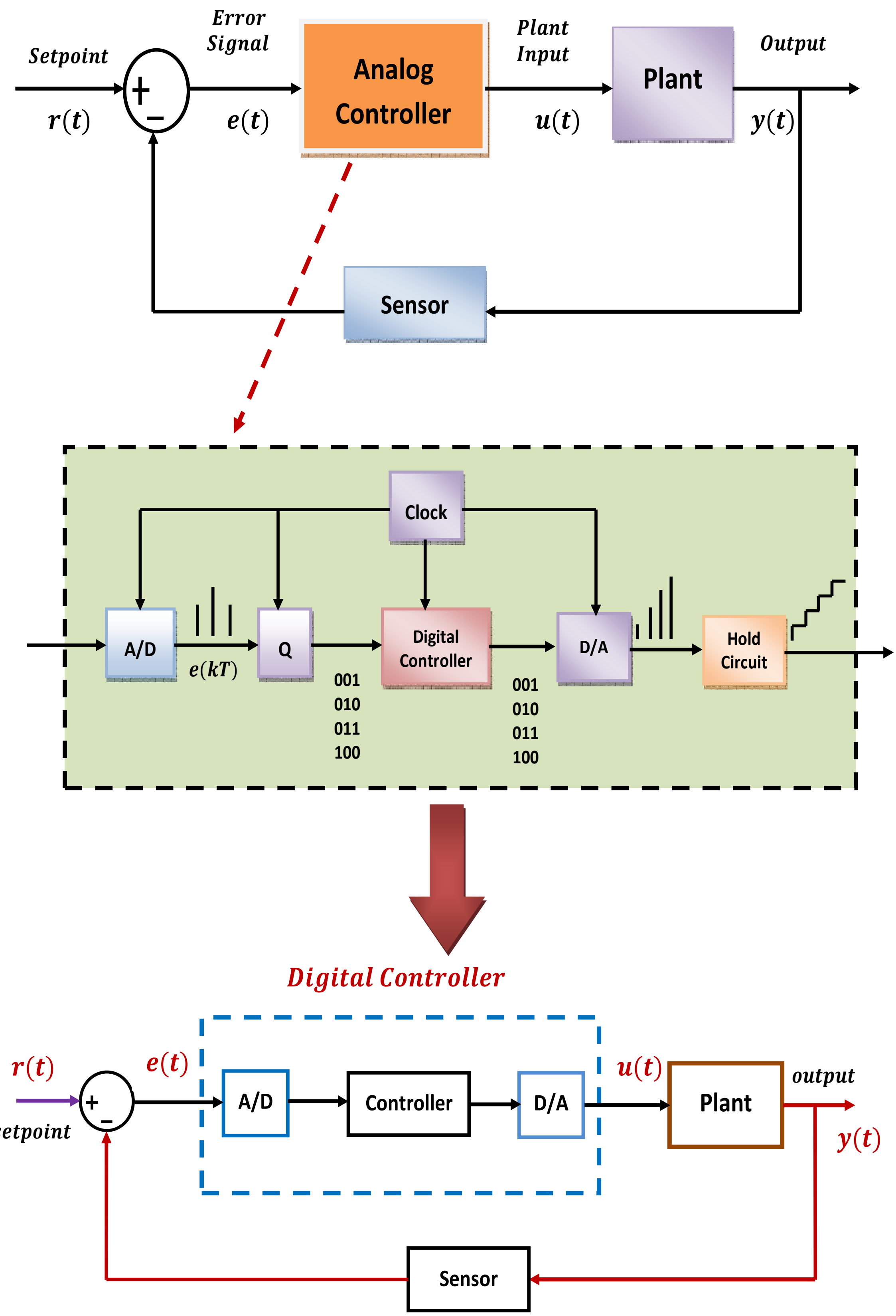

Fig.1.10 Typical block diagram of discrete time feedback control system

5.1. Description of Basic Components of Digital Control system

The operation of the digital control system which is represented by the above typical block diagram is fully accomplished through the following three (03) functional blocks:

- **Analog to Digital Converter (A/D)**:
  This block consists of an electronic device that converts the continuous time (analog) signal to a digital (binary) signal in order to be compatible with the digital controller (micro-processor or microcontroller or any digital-like device) operation. Because, for the computer to respond to any outside event, the data representing this event must be converted into digital form (**1 to 5 volts**), which is being suitably processed and wired to the computer's processor [4].
- **Digital to Analog Converter (D/A)**:
  Digital to Analog Converter (D/A) is an electronic device responsible of converting the output of the digital controller (**pattern of 0s and 1s**) into analog (continuous time) signal to be compatible with the "plant" input (continuous time signal).
- **Digital controller**:
  is the heart of the digital control system, which is an algorithm (software program) implemented to process digital data and generates the appropriate control signal applied at the input of the controlled plant.

The content of the subsequent chapters will be based on the above typical and general structure of discrete time feedback control system.

5.2. Digital Control Systems vs. Analog Control System

Regarding discrete time (digital) control systems, the following advantages over continuous time (analog) control systems can be pointed out:

- As the hardware implementation of analog control systems is a circuit composed of passive and active electronics devices where their properties are highly affected by external factors such as temperature. Hence, the performance of the control system is strongly influenced. However, digital control systems are software based implemented, which means that their dynamic performances are far from the influence of such external affecting factors.

- Since digital control systems are software implemented, consequently, their size is too smaller compared with that of analog control system.
- One of the most important property and advantage of a digital control system over the analog control system consists in its high reproducibility to fulfill the targeted application requirements and specifications. This is because there is an unlimited means of programming. This greatest advantage makes the digital technology more flexible in case of any required modification
- Another valuable advantage of digital control systems consists in their ease of troubleshooting the faults and defected operating conditions compared to the analog control system for which the troubleshooting process is more difficult and laborious.

Despite these great advantages of implementing discrete time (digital) control systems, they however presents some disadvantages such as:

- The mathematical analysis and design of a discrete-time control system is more complex and tedious as compared to continuous-time control system development. This is because of the additional analysis and design parameter, particularly that of the sampling period.
- Because the A/D converter, D/A converter, and the digital computer in reality delay the control signal input (sampling period is not zero), the performance objectives can be more difficult to achieve since the theoretical design approaches usually do not model this small delay.