<u>Exercise 9:</u>

Consider a longer cylinder of radius *R* and length h uniformly charged in volume with a density $\rho > 0$.

1- Find the electric field and potential at any point in space, we take V(0)=0V.

2- Plot the graphs of these two quantities.

<u>Solution</u>

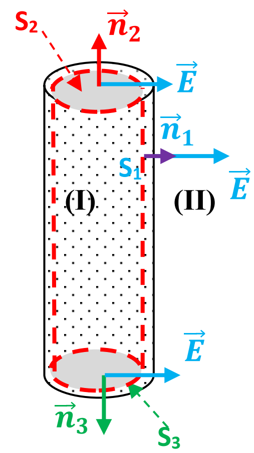
The electric field under these conditions is radial. The Gaussian surface in this case is a cylinder of radius r and height h, with its axis coinciding with the axis of the charged cylinder.

- In this case, two regions are distinguished.

- Region (I); r ≤ R

a)- Find the flux through the Gaussian surface S_G

The total flux through the Gaussian surface:



$$\Phi_{S_G} = \Phi_1 + \Phi_2 + \Phi_3$$

$$\oiint_{S_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}.d\vec{s}_1 + \iint_{S_2} \vec{E}.d\vec{s}_2 + \iint_{S_3} \vec{E}.d\vec{s}_3$$

We have $\vec{E} \perp \vec{s}_2$ $(\vec{E} \perp \vec{n}_2)$ et $\vec{E} \perp \vec{s}_3 (\vec{E} \perp \vec{n}_3)$, since the field is radial, then :

$$\iint_{S_2} \vec{E}_2.d\vec{s}_2 = \iint_{S_3} \vec{E}_3.d\vec{s}_3 = 0$$

with $\vec{E} \parallel d\vec{s}_1$ and in the same direction, then :

$$\underbrace{\vec{E}.\vec{n}_1}_{E\,\|n_1\|\cos 0}\; ds_1$$

$$\oiint_{S_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}.d\vec{s}_1 = \iint_{S_1} E\, ds_1$$

So, we obtain:

Constant on the surface S_1 because *E(r)*

$$\oiint_{S_G} \vec{E}.d\vec{s} = E \iint_{S_1} ds_1$$

S_1 is the lateral surface of the cylinder S_G

$$\oiint_{S_G} \vec{E}.d\vec{s} = E\,S_1$$

Hence:

$$\oiint_{S_G} \vec{E}.d\vec{s} = E\,\underbrace{2\pi\, r\, h}_{S_1} \qquad \textbf{(1)}$$

b)- Find the charge inside the Gaussian surface:
The charge is uniformly distributed over the volume bounded by the **Gaussian cylinder**.

$$Q_{net} = \iiint_{v_{chargé}} \overset{Cte}{\widetilde{\rho}}\; dv$$

Volume bounded by the Gaussian surface S_G

$$Q_{int} = \rho \iiint_{v_{chargé}} dv \;\Longrightarrow\; Q_{net} = \rho v_{chargé}$$

$$Q_{int} = \rho h \pi\, r^2 \qquad (2)$$

From equations (1) and (2), the Gauss's law equality gives us:

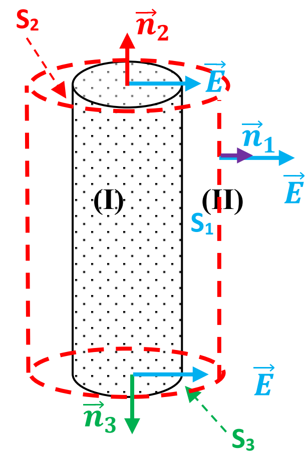
$$E\; 2\pi\, r\, h = \frac{\pi\rho\; h\, r^2}{\varepsilon_0} \qquad \Longrightarrow \qquad \boxed{E\;(r) = \frac{\rho}{2\varepsilon_0}\, r}$$

- Region (II) ; r > R
The expression of the flux through the Gaussian surface is the same as that of the first region.

$$\oiint_{S_G} \vec{E}.d\vec{s} = E\,\underbrace{2\pi\, r\, h}_{S_1} \qquad \textbf{(1)}$$

b)- Calculate the charge inside the Gaussian surface:
The charge is uniformly distributed over the volume bounded by the real cylinder.

$$Q_{int} = \iiint_{v_{chargé}} \overbrace{\rho}^{Cte} \ dv$$



Volume of loaded cylinder

$$Q_{int} = \rho \iiint_{v_{chargé}} dv \ \Longrightarrow Q_{int} = \rho v_{chargé}$$

$$\boldsymbol{Q_{int} = \rho h \pi R^2} \qquad \textbf{(3)}$$

From equations (1) and (3), the Gauss's equality yields:

$$E\ 2\pi\, r\, h = \frac{\pi\rho\, h\, R^2}{\varepsilon_0} \qquad \Longrightarrow \qquad \boxed{E\ (r) = \frac{\rho}{2\varepsilon_0}\left(\frac{R^2}{r}\right)}$$

Conclusion :

The field created by the uniformly charged infinite cylinder in volume is given by:

$$\boxed{E\ (r) = \begin{cases} \dfrac{\rho}{2\varepsilon_0}\, r & 0 \le r \le R \\[2ex] \dfrac{\rho}{2\varepsilon_0}\left(\dfrac{R^2}{r}\right) & ;\ r > R \end{cases}}$$

- Find the electric potential V(r) created by this charge distribution at any point in space:

We know that: $\vec{E} = -\overrightarrow{grad}\, V$

The electric field in this case is radial, so: $\vec{E} = E\ (r)\, \vec{u}_r$

$$E(r) = -\frac{dV}{dr} \quad \Longrightarrow \quad dV = -E(r)\, dr$$

- Region (**I**) ; **r ≤ R** :

We substitute the expression of the electric field in this region

$$dV = -E(r)\, dr \;\; \Rightarrow dV = -\frac{\rho}{2\varepsilon_0}\, r\, dr$$

$$\Rightarrow \;\; V(r) = -\frac{\rho}{4\varepsilon_0}\, r^2 + C_1$$

The constant of integration to be determined

-Region (**II**) ; **r** >**R** :

$$dV = -E(r)\, dr \;\; \Rightarrow dV = -\frac{\rho}{2\varepsilon_0}\left(\frac{R^{\,2}}{r}\right) dr$$

$$\Rightarrow \int \; dV = -\overbrace{\frac{\rho\, R^{\,2}}{2\varepsilon_0}}^{Cte} \int \; \left(\frac{1}{r}\right) dr$$

The constant of integration to be determined

$$\Rightarrow \; V(r) = -\frac{\rho\, R^{\,2}}{2\varepsilon_0}\, ln\,(r) + C_2$$

So, the electric potential created by this charge distribution is:

$$V\,(r) = \begin{cases} -\dfrac{\rho}{4\varepsilon_0}\, r^2 + C_1 & ; 0 \le r \le R \\ \\ -\dfrac{\rho\, R^{\,2}}{2\varepsilon_0}\, ln\,(r) + C_2 & ;\; r > R \end{cases}$$

Now, we need to calculate the integration constants C_1 *and* C_2

According to the exercise data $V(0) = 0$, we notice that this data belongs to region

We calculate first C_1 : pour $r = 0 \;\rightarrow\; V(0) = 0$

$$V(0) = -\frac{\rho}{4\varepsilon_0}\, 0^2 + C_1 \Rightarrow \quad C_1 = 0$$

Hence, the electric potential in region I is:

$$V(r) = -\frac{\rho}{4\varepsilon_0}\, r^2$$

To calculate C_2, we use the continuity of the electric potential at r = R between region I and II.

$$V_{region\,II}(R) = V_{regionI}(R)$$

$$-\frac{\rho\, R^{\,2}}{2\varepsilon_0}\, ln\,(R) + C_2 = -\frac{\rho}{4\varepsilon_0}\, R^2 \Rightarrow C_2 = \frac{\rho\, R^{\,2}}{2\varepsilon_0}\left(\, ln\,(R) - \frac{1}{2}\right)$$

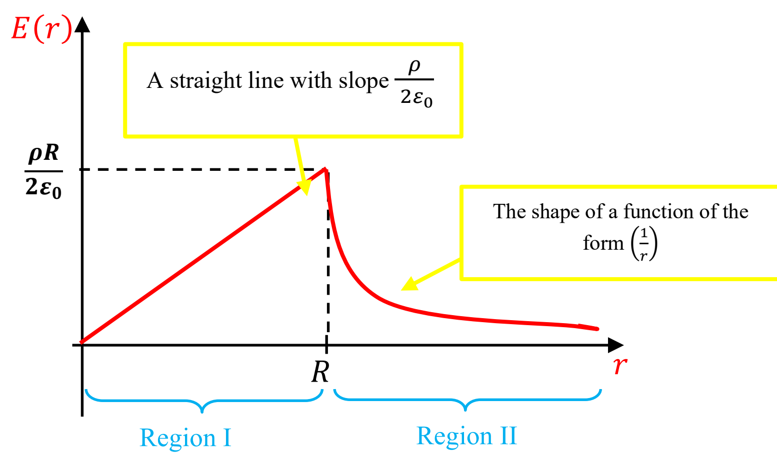
Hence, the electric potential in region II is:

$$V(r) = \frac{\rho\, R^{\,2}}{2\varepsilon_0}\left(ln\ \left(\frac{R}{r}\right) - \frac{1}{2}\right)$$

So, the electric potential created by this charge distribution is:

$$\boxed{V\ (r) = \begin{cases} -\dfrac{\rho}{4\varepsilon_0}\, r^2 & ; \quad 0 \le r \le R \\ \dfrac{\rho\, R^{\,2}}{2\varepsilon_0}\left(ln\ \left(\dfrac{R}{r}\right) - \dfrac{1}{2}\right) & ;\ r > R \end{cases}}$$

- The graphs of $E(r)$



The graphs of $V(r)$