Chapter II
Conductors in electrostatic equilibrium

Course 3

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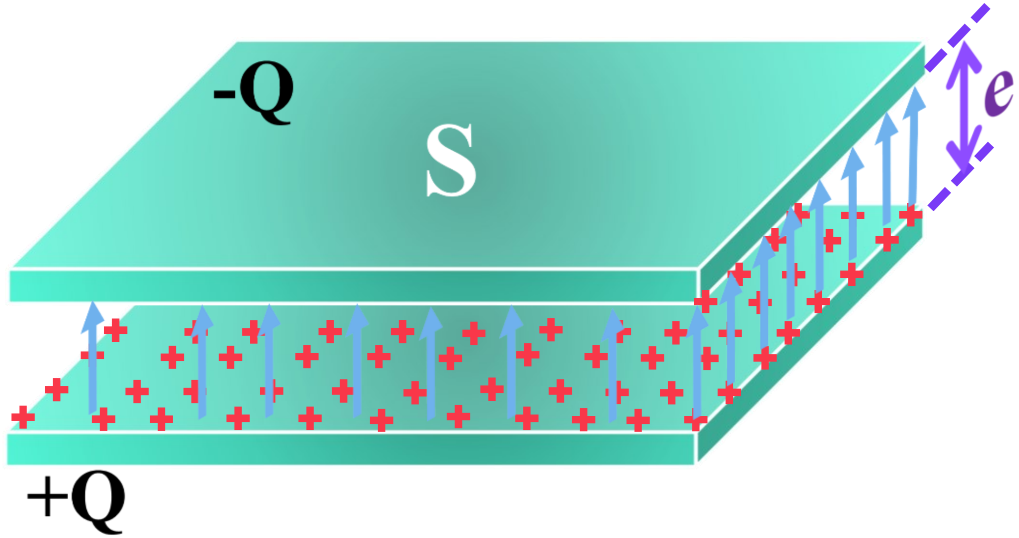
b/ Planar capacitor

It consists of two parallel planar conductors (two plates) facing each other with surface area *S* and thickness *e*, where (e<< S). See Figure below.

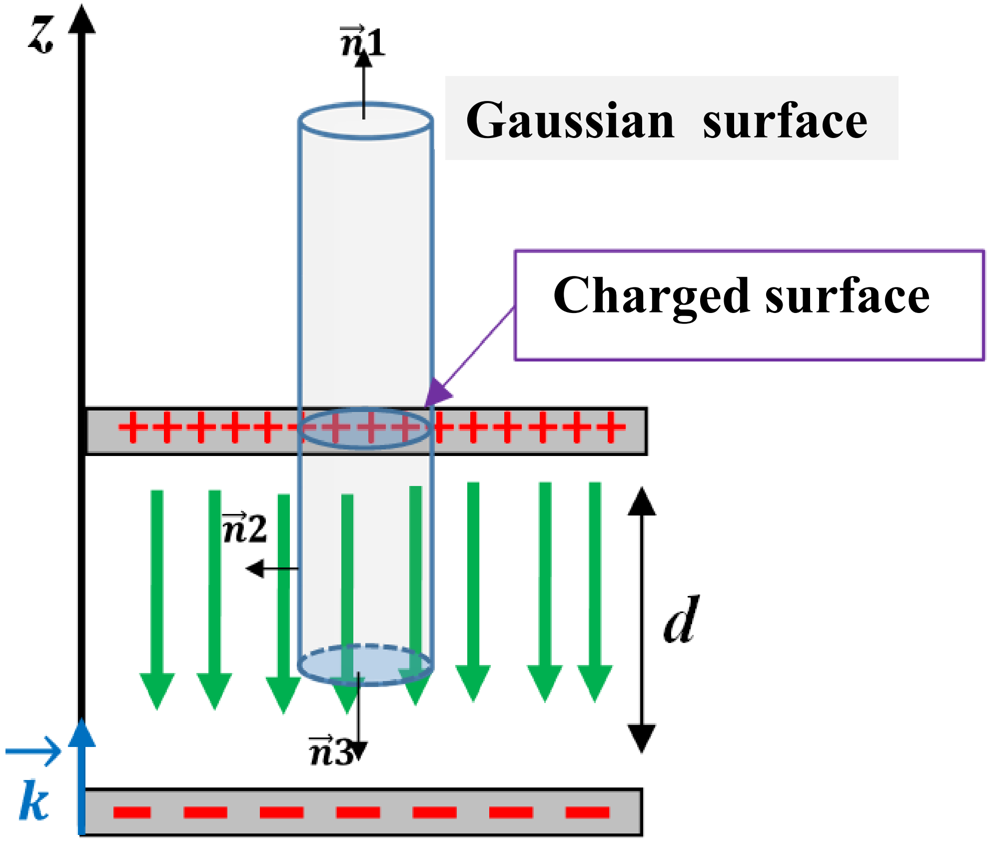


Determine the expression for the capacitance of the planar capacitor:

- Expression of the electric field inside the capacitor:

The electric field $\vec{E}$ in this case is uniform, directed from the positive plate to the negative plate; as shown in the figure below. We can determine the expression for $\vec{E}$ by employing Gauss's theorem.

The Gaussian surface is a cylinder with radius r and height h.



We calculate the flux through the Gaussian surface.

$$\Phi = \oiint_{S_G} (\vec{E}.\overrightarrow{n_1}\mathrm{ds}_1 + \vec{E}.\overrightarrow{n_2}\mathrm{ds}_2 + \vec{E}.\overrightarrow{n_3}\mathrm{ds}_3) = \frac{\sum Q_{int}}{\varepsilon_0}$$

According to the figure, we can deduce that: $\Rightarrow \begin{cases} \vec{E}.\vec{n}_1 = 0\ because \quad E = 0 \\ \vec{E}.\vec{n}_2 = 0\ because \quad \vec{E} \perp \vec{n}_2 \\ \vec{E}.\vec{n}_3 = E\ because \quad \vec{E} \parallel \vec{n}_3 \end{cases}$

then: $\Phi = \iint_{S_1} E.ds_3 = E.S_3 \quad \Rightarrow \quad \Phi = E\pi r^2$

Inside the Gaussian surface, the charge is carried by the disc that intersects the surface of the plate (surface distribution): $Q_{int} = \sigma.S = \sigma\,\pi r^2$

The Gauss's Law equality yields:

$$\Phi = E\pi r^2 = \frac{\sigma\pi\, r^2}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

In vector notation, this would be expressed as:

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\vec{k}$$

- Find the expression of $\Delta V = \big(V(+) - V(-)\big)$

  We know that: $\vec{E} = -\overrightarrow{grad}V$

In Cartesian coordinates, this is expressed as: $\vec{E} = -\overrightarrow{grad}V$

$$E\vec{k} = -\left(\frac{\partial V}{\partial x}\vec{\imath} + \frac{\partial V}{\partial y}\vec{\jmath} + \frac{\partial V}{\partial z}\vec{k}\right) \Rightarrow E\vec{k} = -\left(\frac{dV}{dz}\right)\vec{k}$$

$$E(z) = -\frac{dV}{dz} \quad \Rightarrow \int_{V-}^{V+} dV = -\int_0^d E(z)dz$$

$$V_+ - V_- = -\int_0^d -\frac{\sigma}{\varepsilon_0}dz$$

$$\Delta V = \frac{\sigma}{\varepsilon_0}\int_0^d dz/$$

$$\Delta V = \frac{\sigma}{\varepsilon_0}d$$

- Find the expression of: $C = \frac{Q}{\Delta V}$

Q represents the total charge carried by the positive plate $Q = \sigma S$, which corresponds to the surface area of the plate.

Hence, the expression for a planar capacitor is given by:

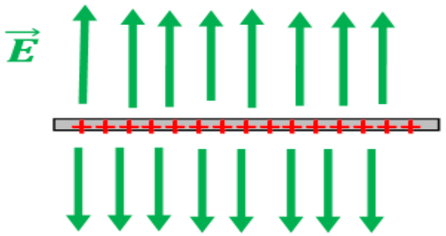
$$C = \frac{\varepsilon_0 S}{d}$$

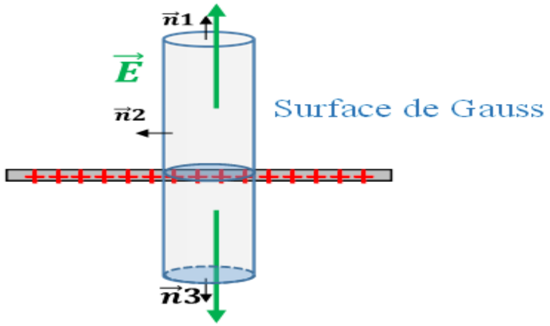
Note: In the following demonstration, we will illustrate the electric field generated by two charged plates—one positively charged and the other negatively charged. This will help clarify why the electric field outside of these plates is zero.

Electric field created by a plane charged with σ >0

It should be noted that the field $\vec{E}$ created by a charged plate (plane) is uniform, as shown in the figure above. The Gaussian surface is a cylinder with a radius of r and a height of h

We calculate the flux through the Gaussian surface.





$$\Phi = \oiint_{S_G} (\vec{E}.\vec{n}_1 ds_1 + \vec{E}.\vec{n}_2 ds_2 + \vec{E}.\vec{n}_3 ds_3) = \frac{\sum Q_{net}}{\varepsilon_0}$$

The vector $\vec{E}$ is parallel to both $\vec{n}_1$ and $\vec{n}_3$ but perpendicular to $\vec{n}_2$.

$$\Longrightarrow \begin{cases} \vec{E}.\vec{n}_1 = E \\ \vec{E}.\vec{n}_2 = 0 \\ \vec{E}.\vec{n}_3 = E \end{cases} \qquad \Longrightarrow \qquad \Phi = \int_{S_1} E.ds_1 + \int_{S_2} E.ds_3 = E.s_1 + E.s_3$$

$\Phi = E\pi r^2 + E\pi r^2 \Longrightarrow \Phi = 2E\pi r^2$

Inside the Gaussian surface, the charge is carried by the disk that intersects the surface of the plate (surface distribution): $Q_{net} = \sigma.S = \sigma.\pi r^2$

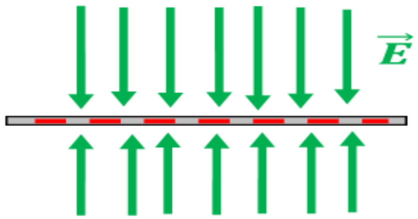
The Gauss's Law equality yields :

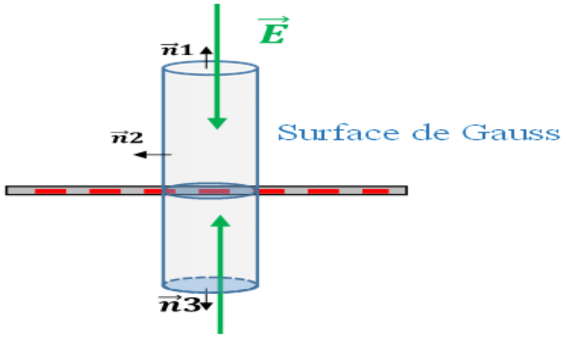
$$\Phi = 2E\pi r^2 = \frac{\sigma.\pi r^2}{\varepsilon_0} \Longrightarrow E = \frac{\sigma}{2\varepsilon_0}$$

The field created by a positive plate is:

$$E_+ = \frac{\sigma}{2\varepsilon_0}$$

Electric field created by a plane charged with σ <0





It should be noted that the field $\vec{E}$ created by a charged plate (plane) is uniform, as shown in the figure above. The Gaussian surface is a cylinder with a radius of r and a height of h

On calcule le flux à travers la surface de Gauss

$$\Phi = \oiint_{S_G} (\vec{E}.\vec{n}_1 ds_1 + \vec{E}.\vec{n}_2 ds_2 + \vec{E}.\vec{n}_3 ds_3) = \frac{\sum Q_{net}}{\varepsilon_0}$$

The vector $\vec{E}$ is antiparallel to both $\vec{n}_1$ and $\vec{n}_3$ but perpendicular to $\vec{n}_2$.

$$\Longrightarrow \begin{cases} \vec{E}.\vec{n}_1 = -E \\ \vec{E}.\vec{n}_2 = 0 \\ \vec{E}.\vec{n}_3 = -E \end{cases} \qquad \Longrightarrow \qquad \Phi = \int_{S_1} E.ds_1 + \int_{S_2} E.ds_3 = -E.s_1 - E.s_3$$

$$\Phi = -E\pi r^2 - E\pi r^2 \Longrightarrow \Phi = -2E\pi r^2$$

Inside the Gaussian surface, the charge is carried by the disk that intersects the surface of the plate (surface distribution): $Q_{net} = -\sigma.S = -\sigma.\pi r^2$

The Gauss's Law equality yields :

$$\Phi = -2E\pi r^2 = -\frac{\sigma.\pi r^2}{\varepsilon_0} \Longrightarrow E = \frac{\sigma.}{2\varepsilon_0}$$

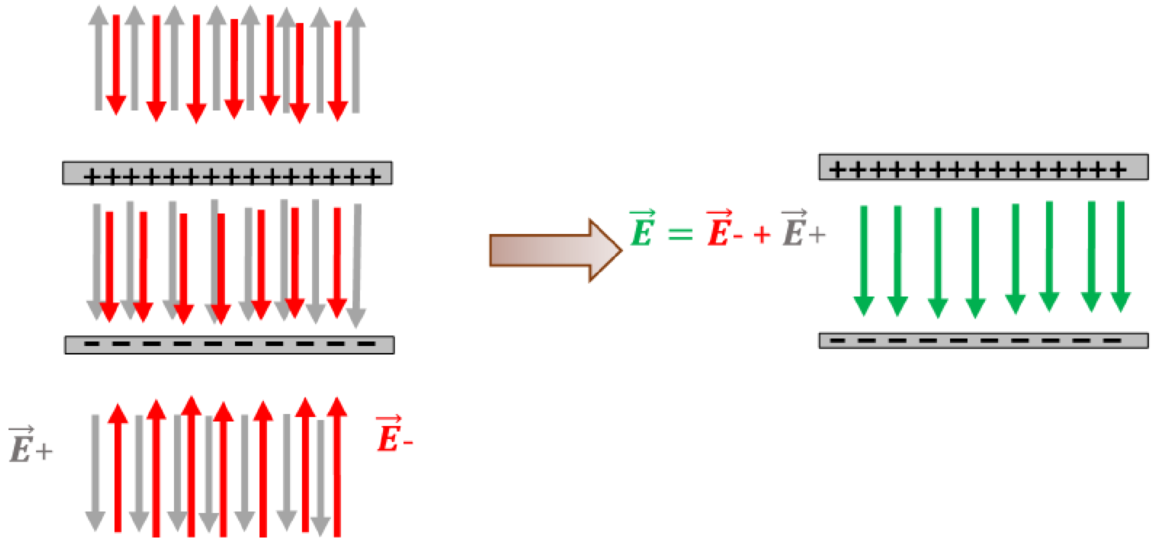
The field created by a negative plate is:

$$\boxed{E_- = \frac{\sigma}{2\varepsilon_0}}$$

Electric field created by the two planes at each point in space:

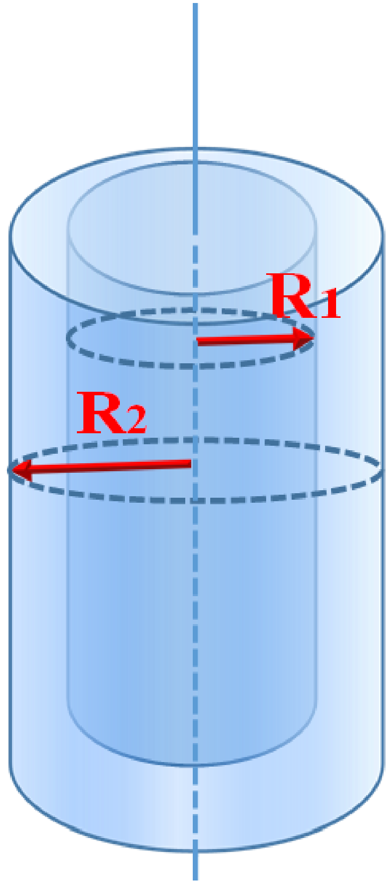
The total field created by the two plates at any point in space:

$$E = \frac{\sigma}{\varepsilon_0}$$



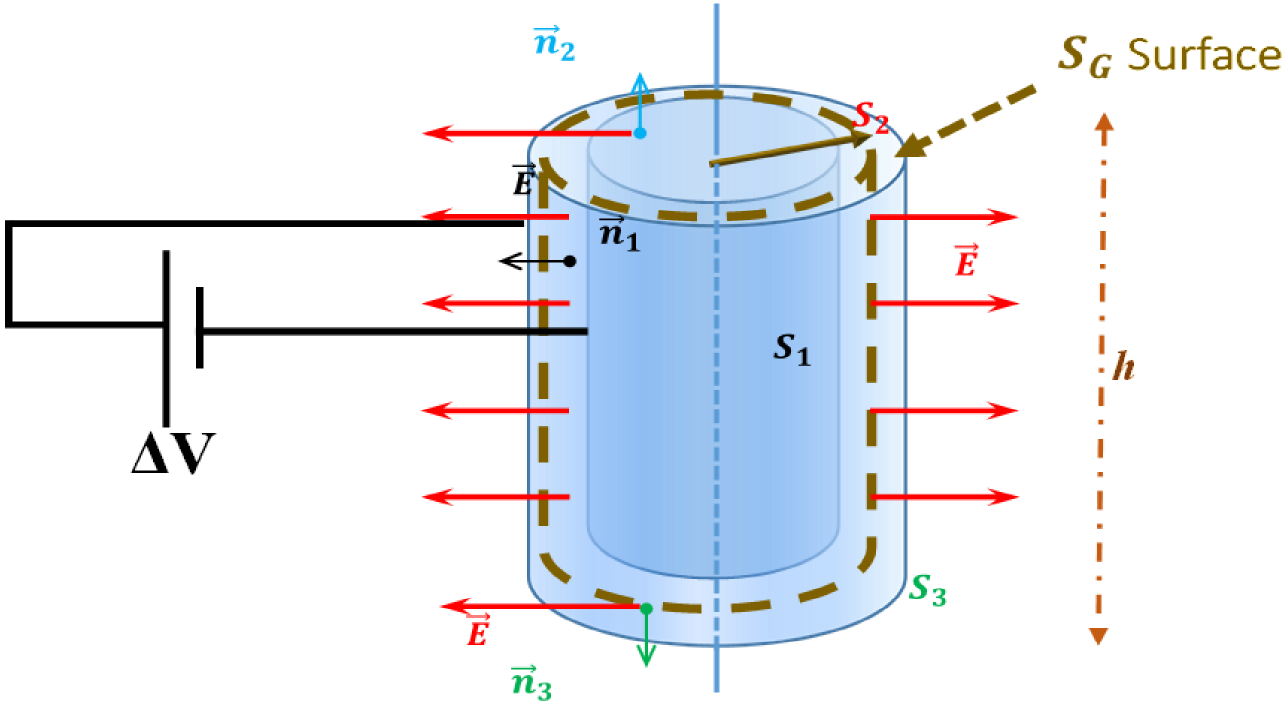
c/ Cylindrical capacitor

The cylindrical capacitor consists of two coaxial conducting cylinders separated by an insulator. See the figure below:



Determine the expression for the capacitance of the cylindrical capacitor:

- Expression of the electric field inside the capacitor:



The electric field inside the capacitor is radial. Therefore, we can apply Gauss's theorem. In this case, the Gaussian surface is a cylinder with height h and radius r, sharing the same axis as the two cylinders.

By applying Gauss's theorem

Flux through the Gaussian surface S_G

$$\Phi = \oint\!\!\oint_{S_G} \vec{E}.d\vec{S} = \frac{Q_{inter}}{\varepsilon_0}$$

The total charge inside the Gaussian surface S_G

- Calculating the flux of the electric field through the surface S_G

The total flux through the Gaussian surface (**closed cylinder S_G**) :

$$\Phi_{S_G} = \Phi_1 + \Phi_2 + \Phi_3$$

$$\oint\!\!\oint_{S_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}\,.d\vec{s}_1 + \iint_{S_2} \vec{E}\,.d\vec{s}_2 + \iint_{S_3} \vec{E}\,.d\vec{s}_3$$

As $\vec{E} \perp \vec{s}_2$ ($\vec{E} \perp \vec{n}_2$) and $\vec{E} \perp \vec{s}_3$ ($\vec{E} \perp \vec{n}_3$) , because the field is radial:

$$\iint_{S_2} \vec{E}\,.d\vec{s}_2 = \iint_{S_3} \vec{E}\,.d\vec{s}_3 = 0$$

$$\underbrace{\vec{E}.\ \vec{n}_1}_{E\,\|n_1\|\cos 0}\ ds_1$$

And as $\vec{E} \parallel d\vec{s}_1$ and in same direction, then :

$$\oint\!\!\oint_{S_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}\,.d\vec{s}_1 = \iint_{S_1} E\ ds_1$$

We obtain:

Constant on the surface S_1 *because* $E(r)$

$$\oint\!\!\oint_{S_G} \vec{E}.d\vec{s} = E\iint_{S_1} ds_1$$

$$\oint\!\!\oint_{S_G} \vec{E}.d\vec{s} = ES_1$$

S_1 is the lateral surface of the cylinder S_G

then: $$\oint\!\!\oint_{S_G} \vec{E}.d\vec{s} = E\,\underbrace{2\pi\, r\, h}_{S_1}$$

The charge inside the Gaussian surface is Q carried by the positive plate: $\sum Q_{net} = Q$

So, Gauss's equality gives us:

$$E 2\pi\, r\, h \;=\; \frac{Q}{\varepsilon_0}$$

$$E(r) = \frac{Q}{\varepsilon_0 2\pi\;\, h} \;\; \frac{1}{r} \quad \text{for:} \;\; \mathrm{R}_1 < r < R_2$$

In vector notation:

$$\vec{E}(r) = \frac{Q}{\varepsilon_0 2\pi\;\, h} \;\; \frac{1}{r} \;\; \overrightarrow{u_r}$$

- Determine the expression $\Delta V = \left(V(+) - V(-)\right)$

We know that: $\vec{E} = -\overrightarrow{grad}V$

In cylindrical coordinates, this gradient is given by:

$$\vec{E} = -\overrightarrow{grad}V = -\left(\frac{\partial V}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial V}{\partial \varphi}\overrightarrow{u_\varphi} + \frac{\partial V}{\partial z}\overrightarrow{u_z}\right)$$

$$\frac{Q}{\varepsilon_0 2\pi\ h}\ \frac{1}{r}\ \overrightarrow{u_r} = -\left(\frac{\partial V}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial V}{\partial \varphi}\overrightarrow{u_\varphi} + \frac{\partial V}{\partial z}\overrightarrow{u_z}\right)$$

$$\Rightarrow E(r) = -\frac{dV}{dr} \quad \Rightarrow \quad \frac{Q}{\varepsilon_0 2\pi\ h}\ \frac{1}{r} = -\frac{dV}{dr}$$

$$\Rightarrow \int_{V_+}^{V_-} dV = -\int_{R_1}^{R_2} \frac{Q}{\varepsilon_0 2\pi\ h}\ \frac{1}{r}\ dr \quad \Rightarrow \quad V_- + V_+ = -\frac{Q}{\varepsilon_0 2\pi\ h}\ \left[ln\, r\right]_{R_1}^{R_2}$$

$$\Rightarrow -\Delta V = -\frac{Q}{\varepsilon_0 2\pi\ h}\ \left[lnR_2 - lnR_1\right] \quad \Rightarrow \quad \Delta V = \frac{Q}{\varepsilon_0 2\pi\ h} ln\left(\frac{R_2}{R_1}\right)$$

- then:

$$C = \frac{Q}{\Delta V}$$

$$C = \frac{Q}{\frac{Q}{\varepsilon_0 2\pi \ h} ln \frac{R_2}{R_1}}$$

Hence, the expression for a cylindrical capacitor is given by:

$$C = \frac{\varepsilon_0 2\pi \ h}{ln \frac{R_2}{R_1}}$$