Chapter I
Electrostatics

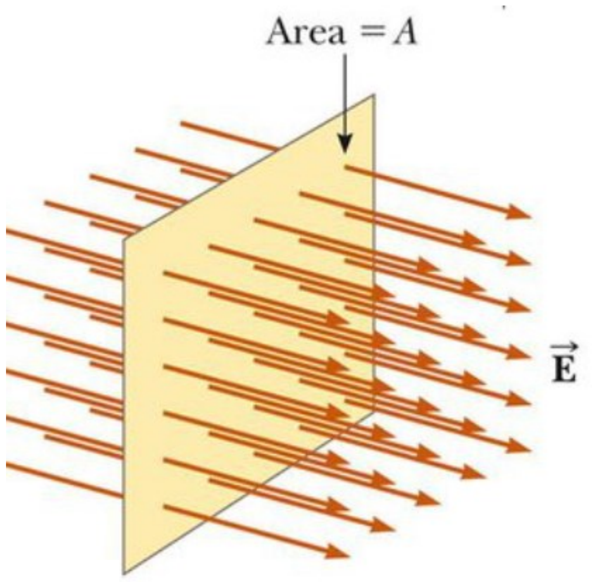
Course 7

Dr IACHACHENE FARIDA   FHC-UMBB-2023-2024

4. Gauss' Law

a. ELECTRIC FLUX

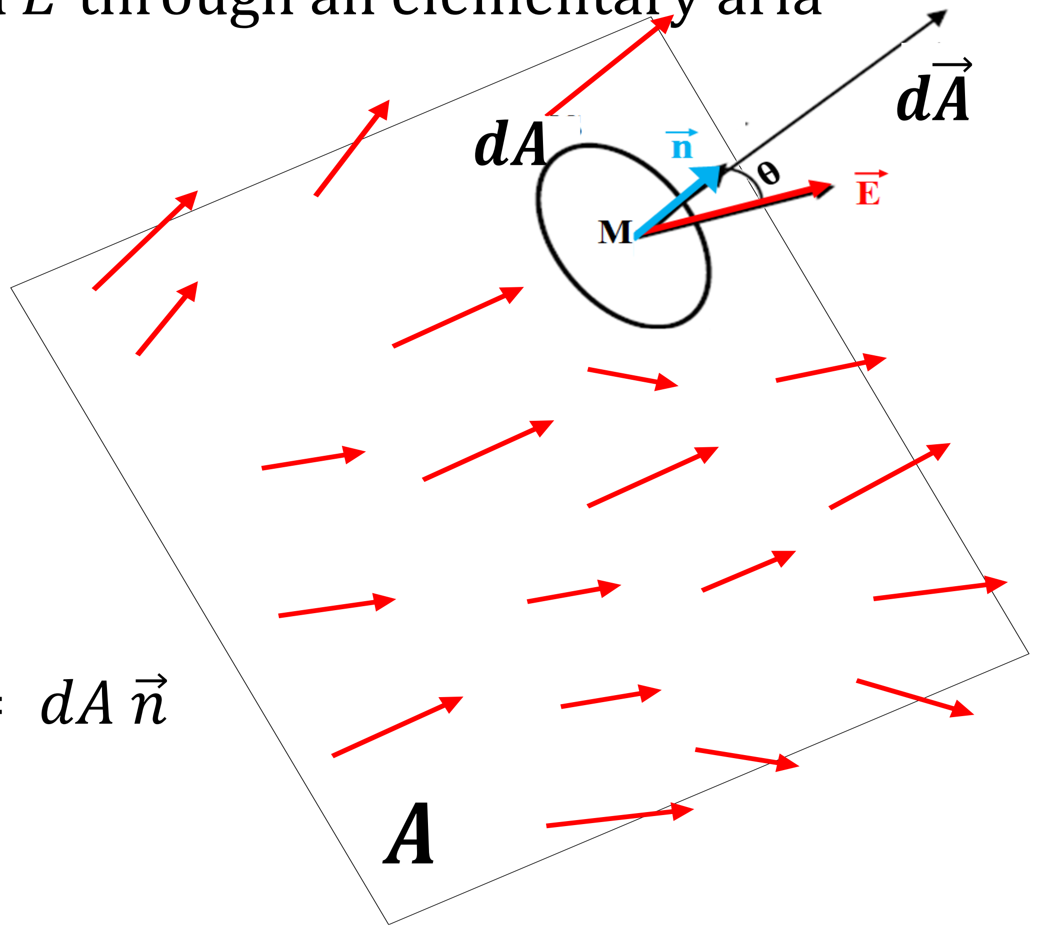
Electric flux is a measure of the electric field passing through a given area. Mathematically, it is defined as the product of the electric field strength and the area perpendicular to the field.



Expression of electrical flux:

We define the elementary flux of the electric field $\vec{E}$ through an elementary aria $d\vec{A}$ by the scalar quantity

$$d\Phi = \vec{E} \cdot d\vec{A}$$



$d\vec{A}$ is a vector normal to aria dA $\left(d\vec{A} \perp dA\right)$

Let $\vec{n}$ be the unit vector carried by $d\vec{A}$,then: $d\vec{A} = dA\,\vec{n}$

$$d\Phi = \vec{E}.\vec{n}\ dA$$

The net flux Φ through a aria A is expressed as the integral

$$\Phi = \int d\Phi = \iint_S \vec{E}.d\vec{A} \qquad \Phi = \iint_S \overbrace{\vec{E}.\vec{n}}^{dot\ product}\ dA$$

Hence $\Phi = \iint_S dA\ \left\|\vec{E}\right\|.\overbrace{\|\vec{n}\|}^{1}\, cos\theta$

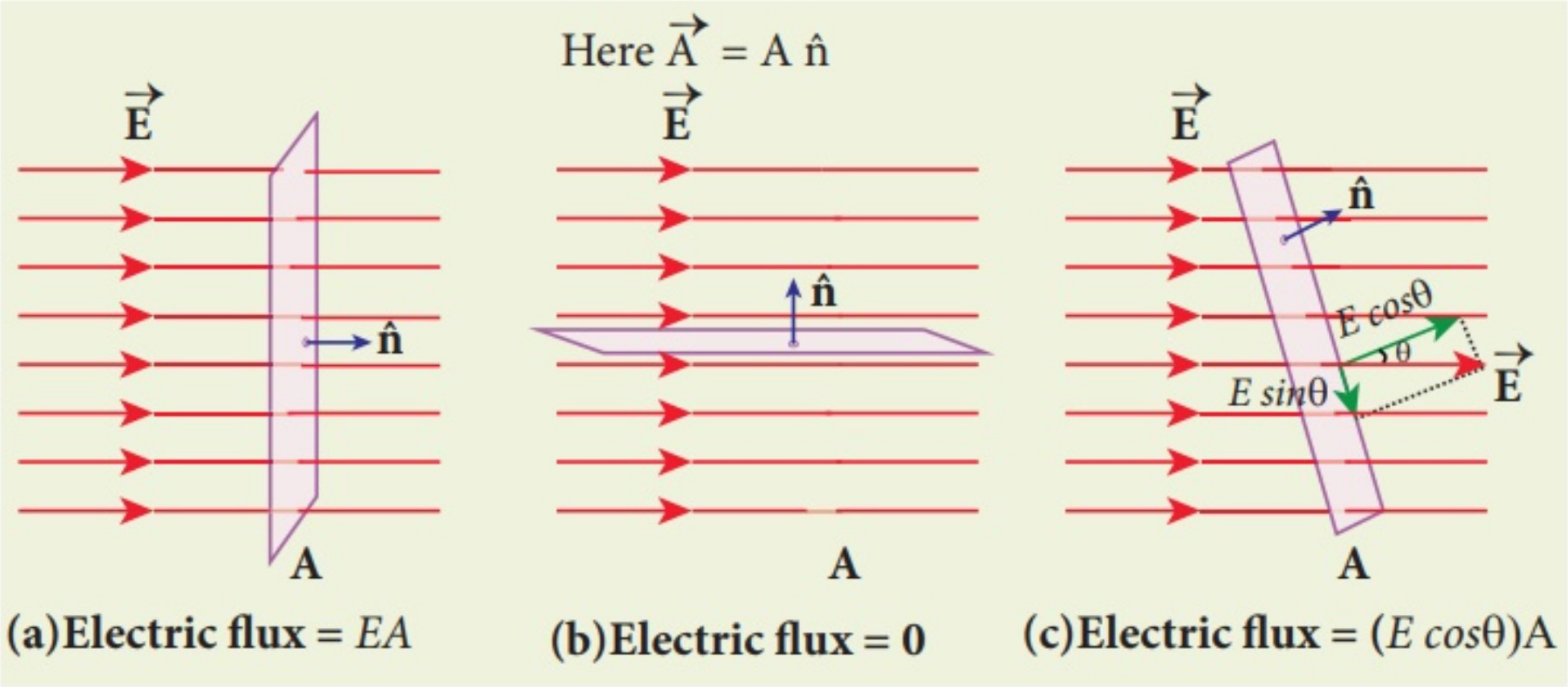
Where θ is the angle between the electric field $\vec{E}$ and the normal to the surface ($\theta = (\vec{E},\vec{n})$)

The unit of Φ in the International System is volt per meter (V·m).

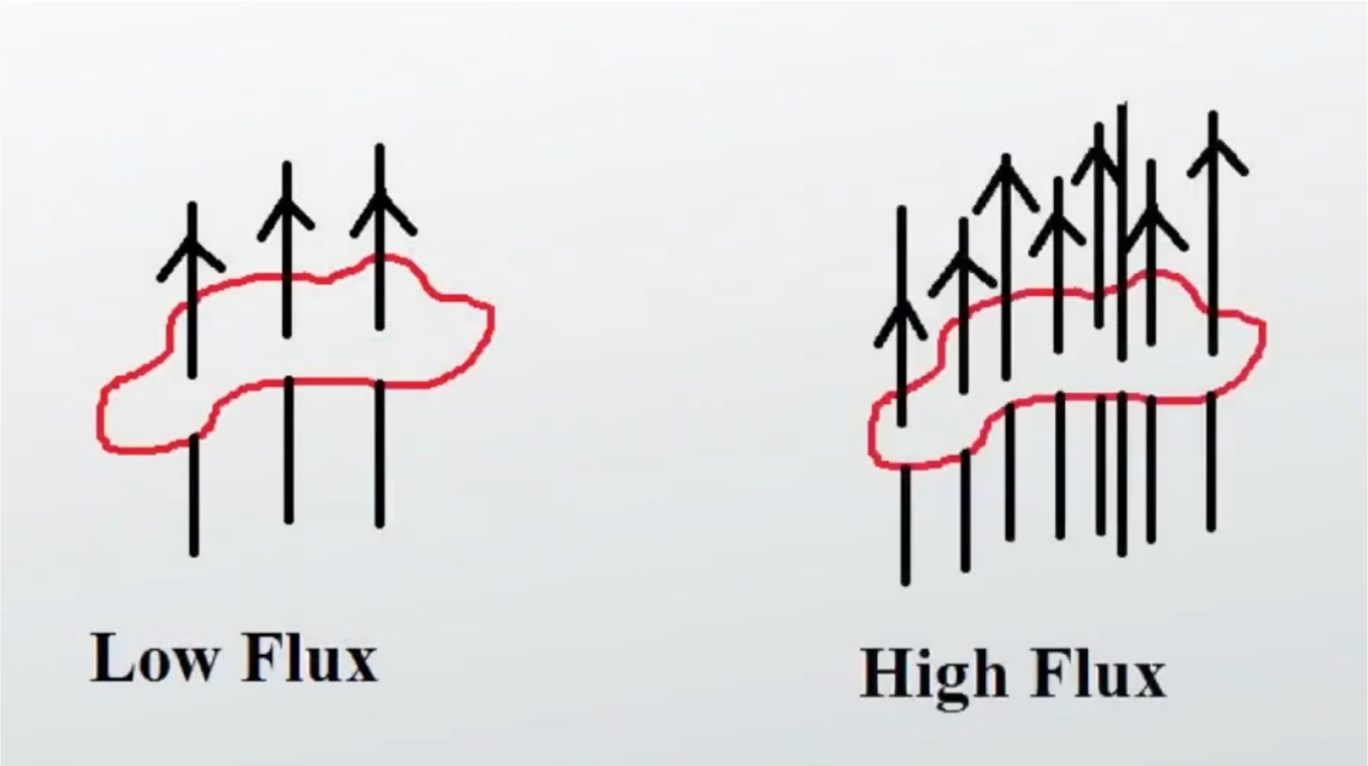
Note : In the case of a closed surface, the unit vector $\vec{n}$ is always oriented from the inside to the outside.

In the case of closed surface: $\Phi = \oiint_{A_G} \vec{E}.d\vec{A}$

Example



The electric flux for Uniform electric field


Low Flux
High Flux

Gauss' Law

Gauss's theorem establishes a relationship between the electric flux through a closed surface and the net charge within the volume bounded by that surface, known as the Gaussian surface (A_G).

Gauss law states that the net electric flux out of a closed surface is equal to the net charge enclosed by that surface divided by the permittivity.

$$\Phi = \oiint_{A_G} \vec{E} . d\vec{A} = \frac{q_{net}}{\varepsilon_0}$$

q_{net} : the net charge is the algebraic sum of all the enclosed positive and negative charges, and it can be positive , negative, or zero

Note : in this course we used: $\Phi = \oiint_{S_G} \vec{E} . d\vec{S} = \frac{q_{net}}{\varepsilon_0}$

Example

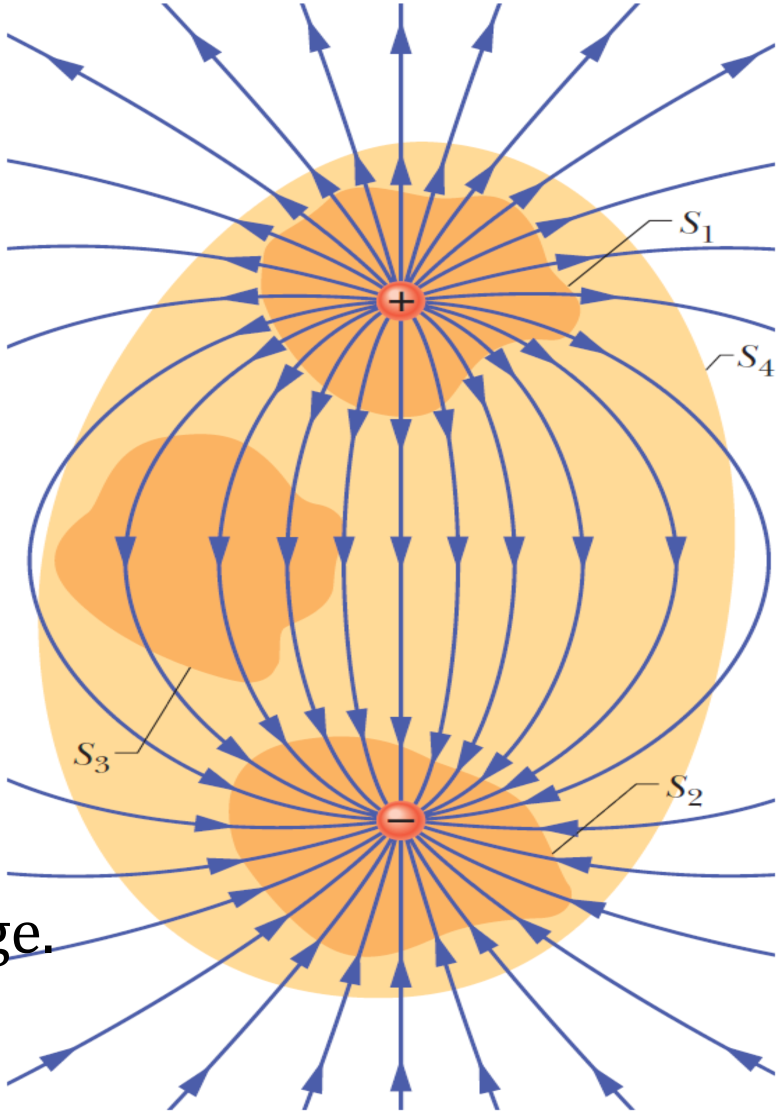
Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

Surface S_1 encloses the positive charge.

Surface S_2 encloses the negative charge.

Surface S_3 encloses no charge.

Surface S_4 encloses both charges and thus no net charge.

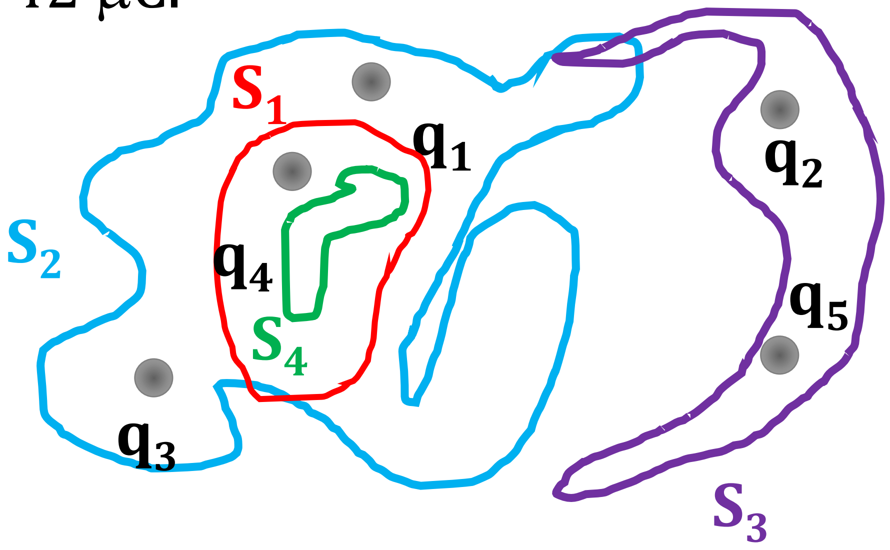


Exemple 2

In space, there exists a distribution of charges and closed surfaces associated with the following charge values (see the figure below):

q_1=+50μC, q_2=-20μC, q_3=+35 μC ,q_4=-15μC et q_5= -42 μC.

and ε_0 =8.845 10^{-12} (IS)



- What are the respective electric fluxes passing through surfaces S_1, S_2, S_3 and S_4 ?

According to Gauss's law, the flux through a closed surface is:

$$\Phi = \oiint_{S_G} \vec{E}.d\vec{s} = \frac{q_{net}}{\varepsilon_0} \rightarrow \Phi = \frac{q_{net}}{\varepsilon_0}$$

The flux through S_1: $\Phi_{S_1} = \frac{q_{net}}{\varepsilon_0} = \frac{q_4}{\varepsilon_0}$, NA: $\Phi_{S_1} = -1.69587\ 10^6$ V.m

The flux through S_2: $\Phi_{S_2} = \frac{q_{net}}{\varepsilon_0} = \frac{(q_4+q_1+q_3)}{\varepsilon_0}$, NA: $\Phi_{S_2} = 7.91407\ 10^6$ V.m

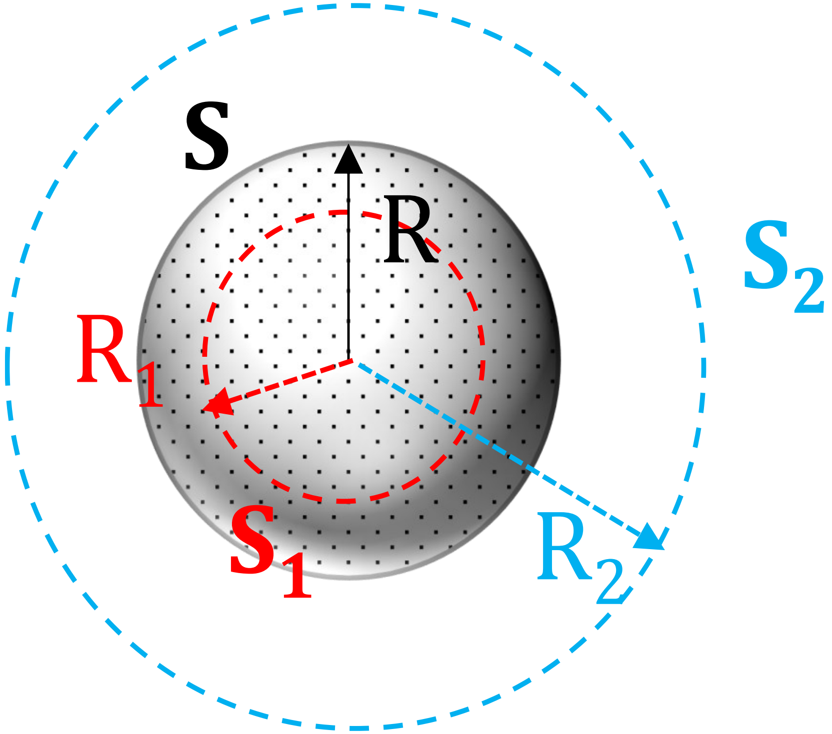
The flux through S_3: $\Phi_{S_3} = \frac{q_{net}}{\varepsilon_0} = \frac{(q_2+q_5)}{\varepsilon_0}$, NA: $\Phi_{S_3} = -7.00961\ 10^6$ V.m

The flux through S_4: $\Phi_{S_4} = \frac{q_{net}}{\varepsilon_0} = \frac{0}{\varepsilon_0}$, NA: $\Phi_{S_4} = 0$ V.m

Exemple 2 :

Let consider a sphere of radius R uniformly charged in volume with a density $\rho > 0$.

What are the respective electric fluxes passing through the imaginary (fictitious, non-real) spherical surfaces S_1 and S_2 of radii R_1 and R_2 respectively, and having the same center as the sphere S (see the figure opposite).



Solution :

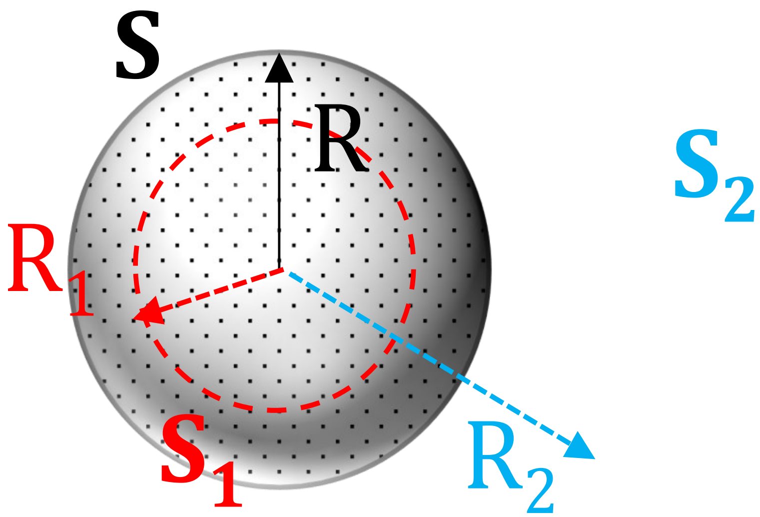
Find expression of electric flux :

1- Throught S_1 :

We know that : $\Phi_{S_1} = \frac{q_{net}}{\varepsilon_0} = \frac{q_{S_1}}{\varepsilon_0}$, Observing the figure,

Note that the charge located within S_1 is the total charge carried by the sphere itself (sphere S_1).

All volume of S_1 is charged by the density ρ



$$q_{S_1} = \iiint_{v_{S_1}} \rho dv = \overbrace{\rho}^{Cte} \iiint_{v_{S_1}} dv = \rho v_{S_1} = \rho \frac{4}{3} \pi R_1^3$$

Hence : $\Phi_{S_1} = \frac{\rho \frac{4}{3} \pi R_1^3}{\varepsilon_0} \quad \Longrightarrow \quad \Phi_{S_1} = \rho \frac{4}{3\varepsilon_0} \pi R_1^3$

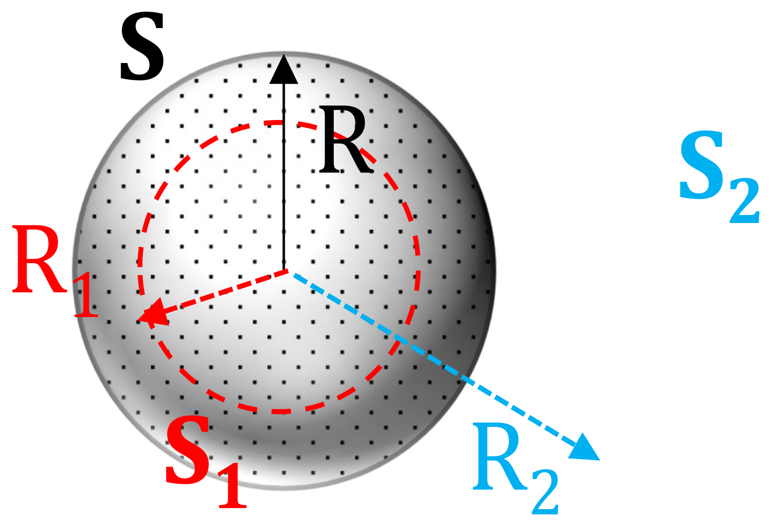
2 - Throught S_2

We khow that : $\Phi_2 = \frac{q_{\text{net}}}{\varepsilon_0} = \frac{q_{S_2}}{\varepsilon_0}$, Observing the figure, the charge inside the sphere S_2 is the total charge carried by the sphere S

Since the charge distribution is volumetric, then:

$$q_{S_2} = q_S = \iiint_{v_s} \rho dv = \overbrace{\rho}^{Cte} \iiint_{v_s} dv = \rho v_S = \rho \frac{4}{3} \pi R^3$$

$$\text{Hence : } \Phi_{S_2} = \frac{\rho \frac{4}{3} \pi R^3}{\varepsilon_0} \quad \Longrightarrow \quad \Phi_{S_2} = \rho \frac{4}{3\varepsilon_0} \pi R^3$$



Gauss' Law Applications

The theorem facilitates the calculation of the electric field in problems with symmetrical characteristics. This method proves simpler compared to direct computation.

In cases where the electric field $\vec{E}$ remains constant across the Gaussian surface, it can be factored out of the integral, leading to:

- when $\theta = (\vec{E}, \vec{n}) = 0 \Longrightarrow \Phi = \oiint_S E\, ds = \frac{q_{net}}{\varepsilon_0} \Longrightarrow E \oiint_S ds = \frac{q_{net}}{\varepsilon_0}$

$$\Longrightarrow \mathrm{E} = \frac{q_{net}}{\varepsilon_0 \oiint_S ds}$$

- when $\theta = (\vec{E}, \vec{n}) = \pi \Longrightarrow \Phi = \oiint_S -E\, ds = \frac{q_{net}}{\varepsilon_0} \Longrightarrow -E \oiint_S ds = \frac{q_{net}}{\varepsilon_0}$

$$\Longrightarrow \mathrm{E} = -\frac{q_{net}}{\varepsilon_0 \oiint_S ds}$$

Problem-Solving Strategy: Gauss's Law

1. Identify the spatial symmetry of the charge distribution. This is an important first step that allows us to choose the appropriate Gaussian surface . As example, infinite line of charge has cylindrical symmetry

2. Choose a Gaussian surface with the same symmetry as the charge distribution and identify its consequences.

3. Evaluate the integral $\oiint_{S_G} \vec{E}.d\vec{S}$ over the Gaussian surface , that is, calculate the flux through the surface.

4. Determine the amount of charge q_{net} enclosed by the Gaussian surface.

5. Evaluate the electric field of the charge distribution. The field may now be found using the results of steps 3 and 4.

Basically, there are only four types of symmetry that allow Gauss's law to be used to deduce the electric field:

- A charge distribution with spherical symmetry.
- A charge distribution with cylindrical symmetry.
- A charge distribution with planar symmetry.
- A charge distribution in infinite line.

To exploit the symmetry, we perform the calculations in appropriate coordinate systems and use the right kind of Gaussian surface for that symmetry, applying the remaining four steps

Examples

Example 01: Field created by an infinite straight wire

Consider an infinite straight wire uniformly charged with a density $\lambda>0$.

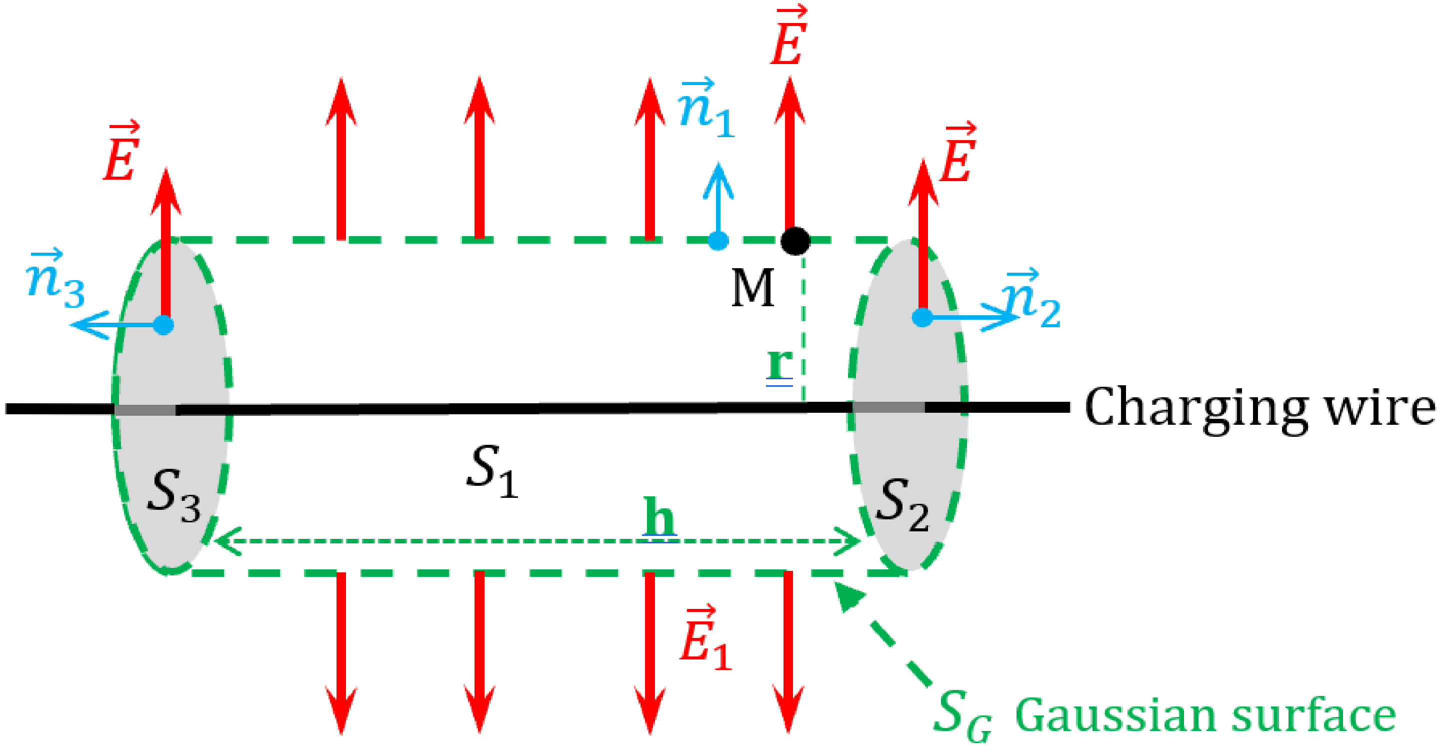
Find the electric field at any point in space created by this wire.

Charging wire

Solution:

Expression of the electric field:

It is noted that the charge distribution remains invariant under rotation around the wire and translation parallel to the wire. In this case, the Gaussian surface S_G is a closed cylinder with its axis coinciding with the charged wire (see the figure). Let h and r be the height and the radius of this cylinder, respectively.



Following Gauss's law: $\Phi = \oiint_{S_G} \vec{E}.d\vec{s} = \frac{q_{\text{net}}}{\varepsilon_0}$

Calculate the flux of the electric field through the surface S_G

The net flux : $\Phi_{S_G} = \Phi_1 + \Phi_2 + \Phi_3$

$$\oiint_{A_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}_1.d\vec{s}_1 + \iint_{S_2} \vec{E}_2\,.d\vec{s}_2 + \iint_{S_3} \vec{E}_3.d\vec{s}_3$$

As $\vec{E}_2 \perp d\vec{s}_2$ et $\vec{E}_3 \perp d\vec{s}_3$ then: $\iint_{S_2} \vec{E}_2.d\vec{s}_2 = \iint_{S_3} \vec{E}_3.d\vec{s}_3 = 0$

And as $\vec{E}_1 \parallel d\vec{s}_1$ and in the sae direction, then :

$$\oiint_{S_G} \vec{E}.d\vec{s} = \iint_{S_1} \vec{E}_1.d\vec{s}_1 = \iint_{S_1} E \quad ds_1$$

Therefore, we obtain: $\boldsymbol{\oiint_{S_G} \vec{E} \cdot d\vec{s} = E_1\, 2\pi\, r\, h}$

Calculate the net charge through S_G

Uniform linear distribution of charge along a wire:

$$dq = \lambda . dl \quad \Rightarrow \quad Q_{net} = \int dq = \lambda \int_{h} dl \quad \Rightarrow \quad \boldsymbol{Q_{net} = \lambda . h}$$

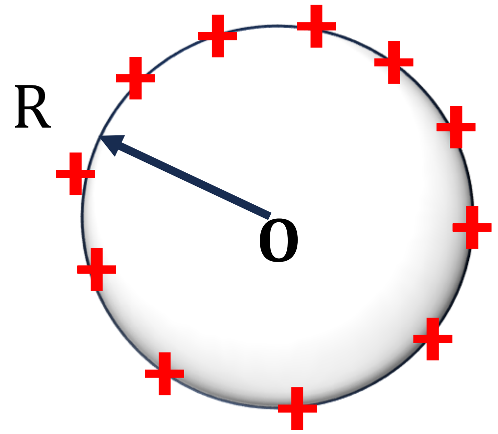
By substituting into Gauss's equation, we obtain:

$$E_1 . 2\pi . r . h = \frac{\lambda . h}{\varepsilon_0} \quad \Rightarrow \quad \boxed{E(r) = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r} = \frac{2K\,\lambda}{r}}$$

Example 02: Electric Field Created by a Uniformly Charged Sphere Surface

Consider a sphere S with center O and radius R, uniformly charged on its surface with a surface charge density $\sigma>0$.

Find the electrostatic field created by this sphere at any point in space.



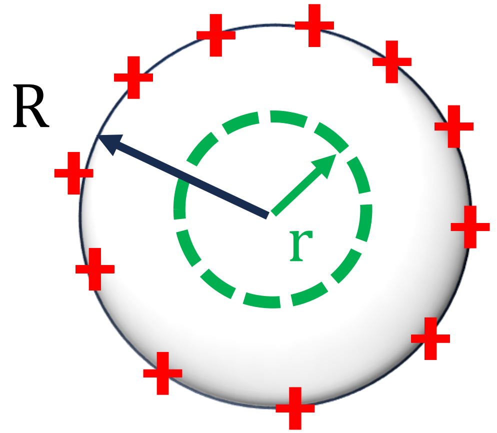
Solution

We notice that the charge distribution is spherically symmetric, so the field is radial. The Gaussian surface S_G in this case is a sphere with center O and radius r.

When r < R (inside the sphere S)

It is clear that inside S_G the charge is zero.

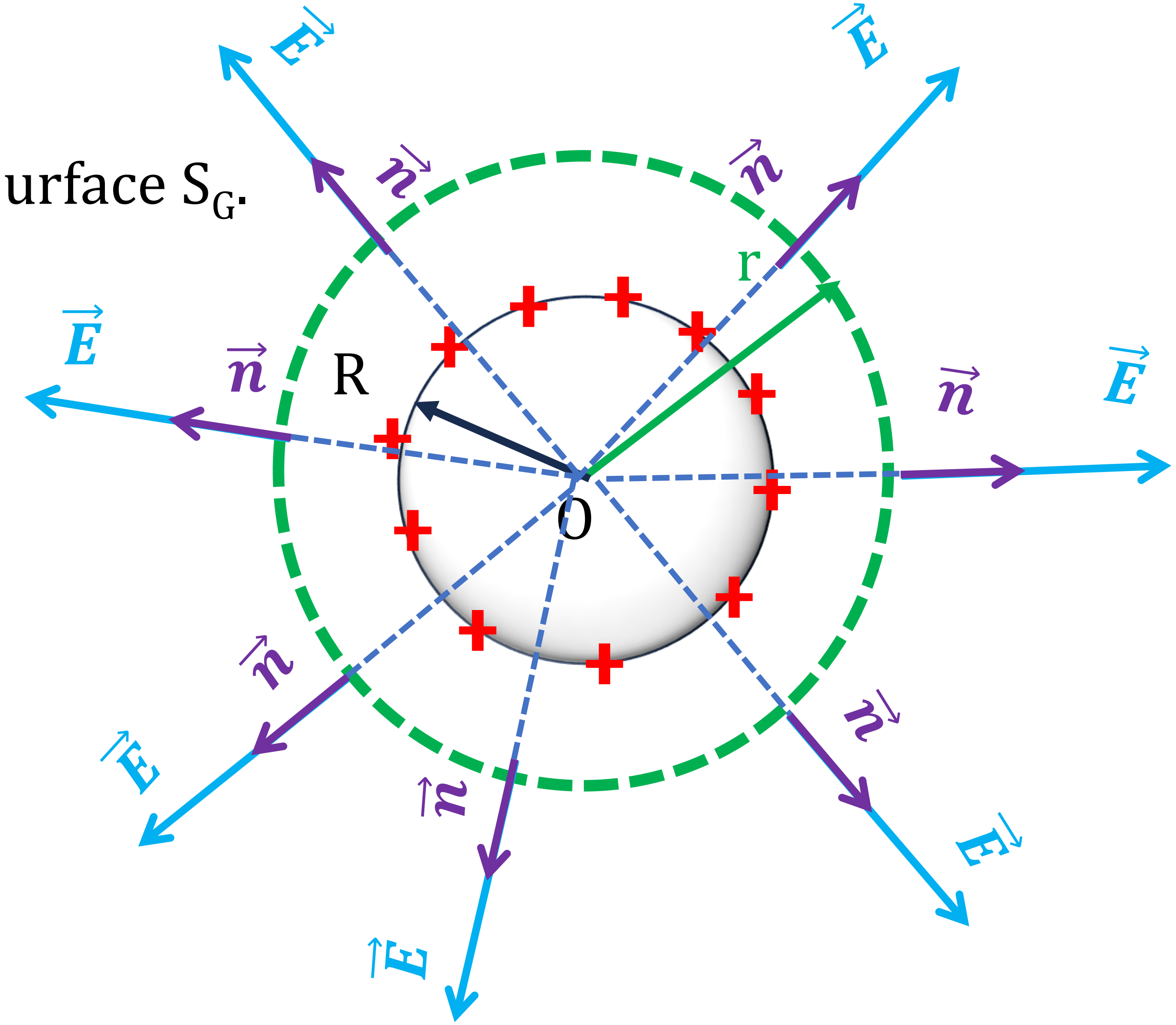
If $Q_{int} = 0 \quad \Longrightarrow \quad E(r) = 0$



- When r > R (outside the sphere S)

Find the flux of the electric field through the surface S_G.

In this case $\vec{E} \mathbin{/\!/} \vec{n}$



$$\Phi_{S_G} = \oiint_{S_G} \vec{E}.d\vec{s} = \oiint_{S_G} \overbrace{\vec{E}\,.\vec{n}}^{E}.ds = E \oiint_{S_G} ds$$

$$\Phi_{S_G} = ES_G \qquad \Longrightarrow \Phi_{S_G} = 4\pi\, r^2 E$$

Determine the charge located inside the Gaussian surface.

It is clear that inside S_G, the charge is distributed on the surface of the charged sphere S.

Hence,

- Surface distribution: $Q_{int} = \iint_S \overset{Cte}{\widetilde{\sigma}} \, ds \implies Q_{int} = \sigma \iint_S \, ds \, Q_{int}$

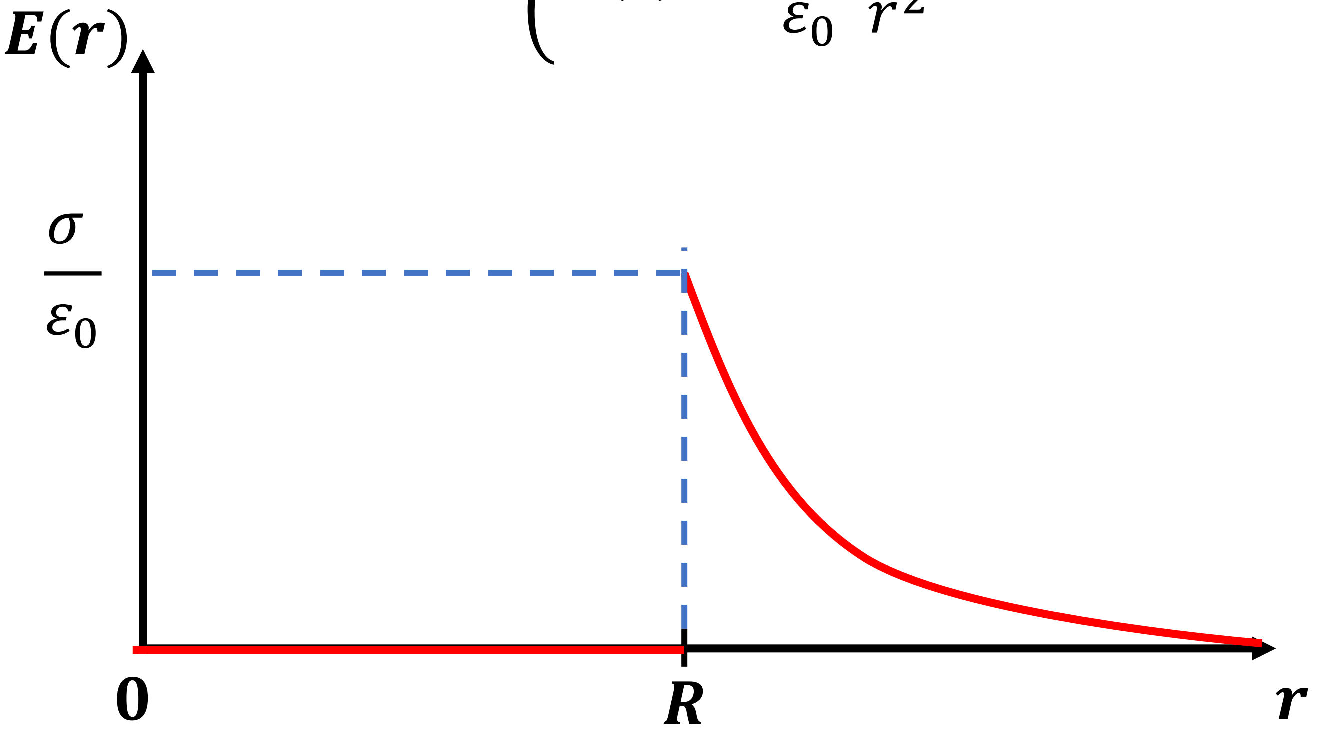
$$Q_{int} = \sigma\, S = 4\pi R^2\, \sigma$$

By substituting into the Gauss's law equation, we obtain:

$$4\pi\, r^2 E = \frac{4\pi R^2\, \sigma}{\varepsilon_0} \implies E = \frac{\sigma}{\varepsilon_0}\frac{R^2}{r^2}$$

So, the field created by a uniformly charged spherical surface is:

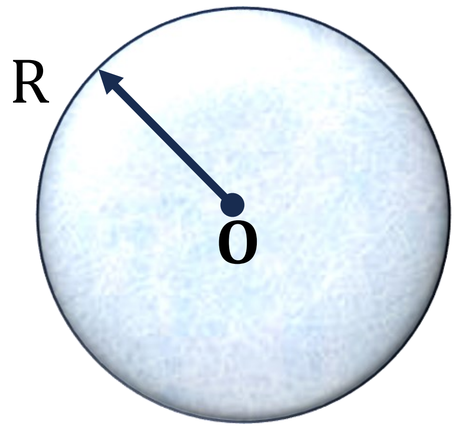
$$E(r)\begin{cases} E(r) = 0 & ,r < R \\ E(r) = \dfrac{\sigma}{\varepsilon_0}\dfrac{R^2}{r^2} & ,r \geq R \end{cases}$$



Example 03: Electric Field Created by Sphere Uniformly Charged in volume

Consider a sphere S with center O and radius R, which has electric charge uniformly distributed throughout its entire volume with a volume charge density $\rho > 0$.

Find the electrostatic field created by this sphere at any point in space.



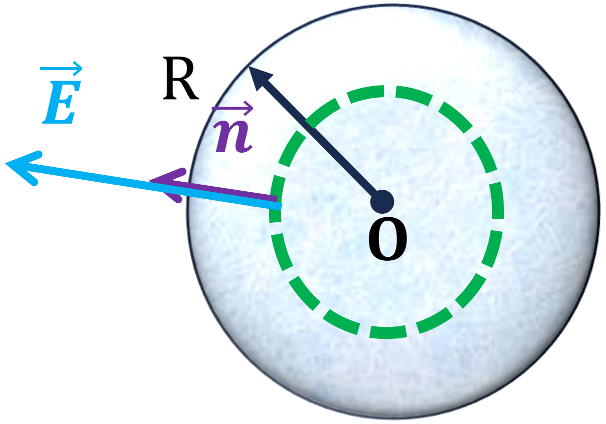
Solution

We notice that the charge distribution is spherically symmetric, so the field is radial. The Gaussian surface S_G in this case is a sphere with center O and radius r.

- When r < R (inside the sphere S)

Find the flux of the electric field through the surface S_G.

In this case $\vec{E} \,/\!/\, \vec{n}$



$$\Phi_{S_G} = \oiint_{S_G} \vec{E}.d\vec{s} = \oiint_{S_G} \overbrace{\vec{E}\,.\vec{n}}^{E}.ds = E\oiint_{S_G} ds$$

$$\Phi_{S_G} = ES_G \qquad \Longrightarrow \Phi_{S_G} = 4\pi\, r^2 E$$

Determine the charge located inside the Gaussian surface.

It is clear that inside S_G, the charge is distributed on all her volume

Hence,

Volume charge distribution: $Q_{int} = \iiint \overbrace{\rho}^{Cte} \, dv \implies Q_{int} = \rho \iiint dv$

$$Q_{int} = \rho\, V = \rho \frac{4}{3} \pi r^3$$

By substituting into the Gauss's law equation, we obtain:

$$4\pi\, r^2 E = \frac{\rho \frac{4}{3} \pi r^3}{\varepsilon_0} \implies E = \frac{\rho}{3\varepsilon_0} r$$

- When r > R (outside the sphere S)

Find the flux of the electric field through the surface S_G.

$$\Longrightarrow \Phi_{S_G} = 4\pi\, r^2 E$$

Determine the charge located inside the Gaussian surface.

It is clear that inside S_G, the charge is distributed on the volume of the charged sphere S.

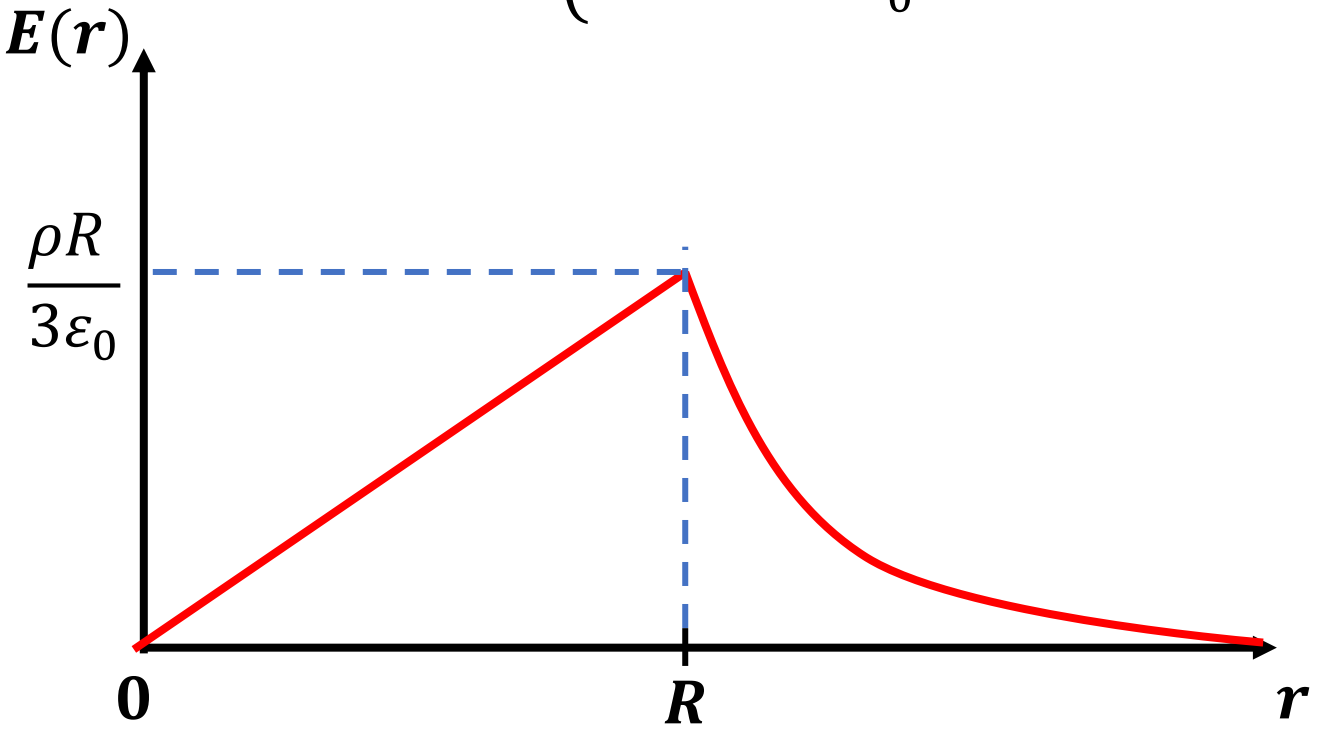
$$Q_{int} = \rho\, V = \rho \frac{4}{3}\pi R^3$$

By substituting into the Gauss's law equation, we obtain:

$$4\pi\, r^2 E = \frac{\rho \frac{4}{3}\pi R^3}{\varepsilon_0} \quad \Longrightarrow \quad E = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}$$

So, the field created by a uniformly charged spherical surface is:

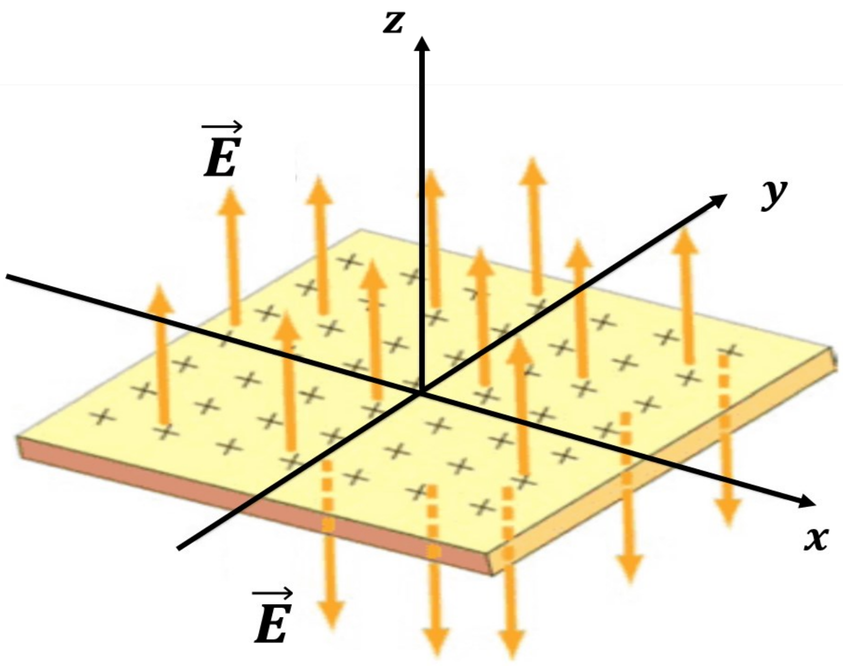
$$E(r)\begin{cases} E(r) = \dfrac{\rho}{3\varepsilon_0} r & , r < R \\ E(r) = \dfrac{\rho}{3\varepsilon_0}\dfrac{R^3}{r^2} & , r \geq R \end{cases}$$



Example 04: Electric Field Created by an Infinitely Thin Plane Uniformly Charged with Positive Charge

Let us consider an infinitely thin plane that is uniformly charged with a positive charge, characterized by density σ.

Find the electrostatic field created by this plan everywhere in space.



Solution

We calculate the flux through the Gaussian surface.

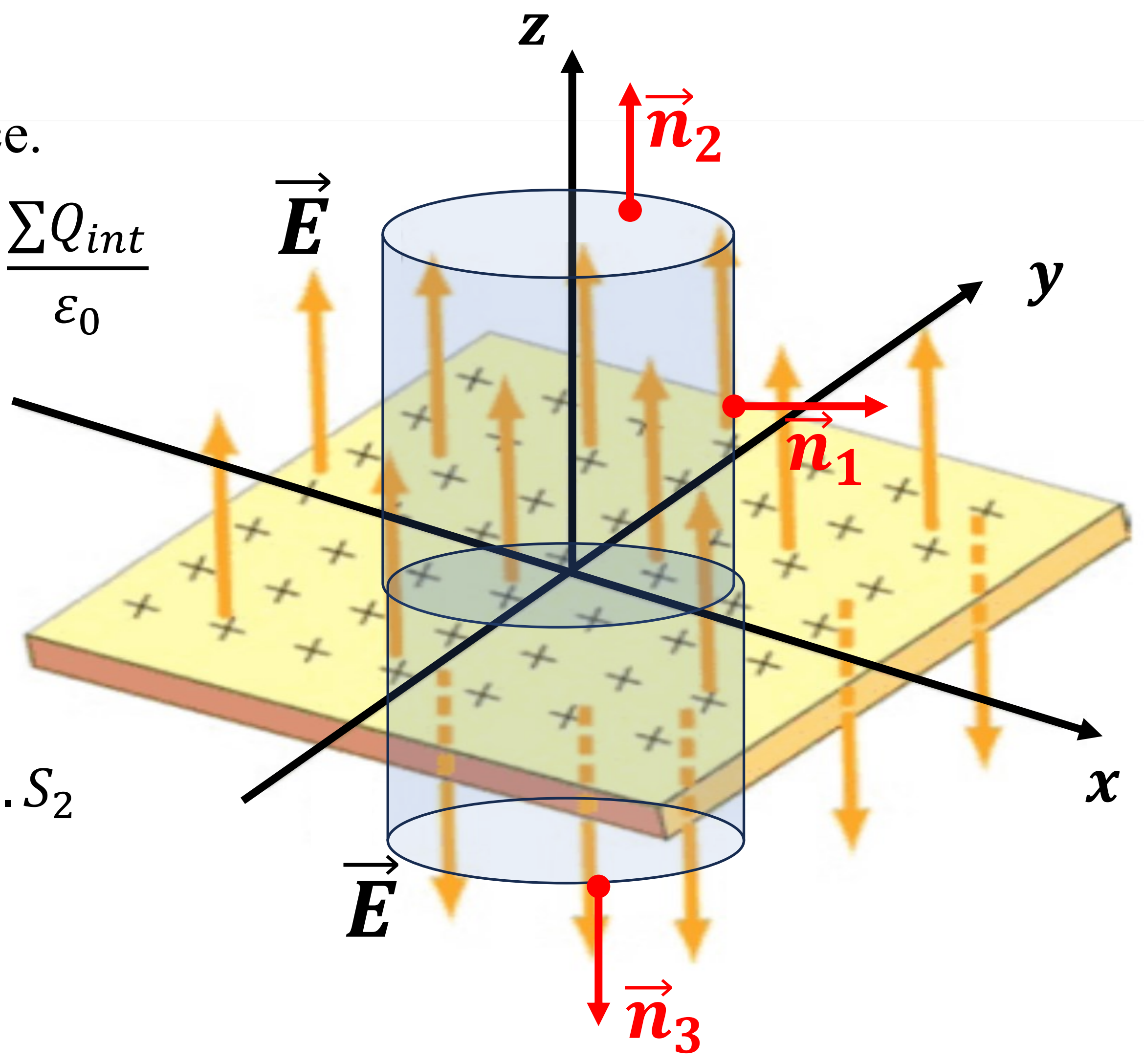
$$\Phi = \oiint_{S_G} (\vec{E}.\overrightarrow{n_1}\mathrm{ds}_1 + \vec{E}.\overrightarrow{n_2}\mathrm{ds}_2 + \vec{E}.\overrightarrow{n_3}\mathrm{ds}_3) = \frac{\sum Q_{int}}{\varepsilon_0}$$

From the figure, we can deduce that:

$$\Rightarrow \begin{cases} \vec{E}.\vec{n}_1 = 0 \quad , \vec{E} \perp \vec{n}_1 \\ \vec{E}.\vec{n}_2 = E \ \ , car\ \vec{E} \parallel \vec{n}_2 \\ \vec{E}.\vec{n}_3 = E\ car\ \vec{E} \parallel \vec{n}_3 \end{cases}$$

D'où : $\Phi = \iint_{S_2} E.ds_2 + \iint_{S_3} E.ds_3 = E.S_3 + E.S_2$

$$\Rightarrow \quad \Phi = 2E\pi r^2$$



Inside the Gaussian surface, the charge is carried by the disk that intersects the surface of the plate (surface distribution).

$$Q_{net} = \sigma . S = \sigma \pi r^2$$

$$\Phi = 2E\pi r^2 = \frac{\sigma\pi\, r^2}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$